自動控制

Chapter 4
理論基礎與參考材料：動態系統的模型化

※ 控制系統中的直流馬達

1. 直流馬達是目前工業界用途最廣的驅動裝置。
2. 稀土族金屬磁鐵的發展，已能做出非常高轉矩-體積比，且價格合理之永磁式直流馬達。
3. 具備非常高的轉矩-慣量比且低時間常數的特性之的直流馬達，已打開在計算機週邊裝置中新的應用，譬如磁帶驅動、印表機、磁碟機驅動、文字處理機，以及工具機工業中之應用 。

※ 直流馬達的基本操作原理

1. 直流馬達基本上是一轉矩轉換器，可將電能轉換成機械能。
2. 馬達軸產生的轉矩直接與場磁及電樞電流成正比。
3. 如圖 4-64 所示，在具有磁通 ϕ 的磁場裡建立一帶電導體，此導體離旋轉中心之距離為 r。
4. 磁通 ϕ、電流 i_a 與產生轉矩之間的關係

$$T_m = K_m \phi i_a \qquad (4\text{-}200)$$

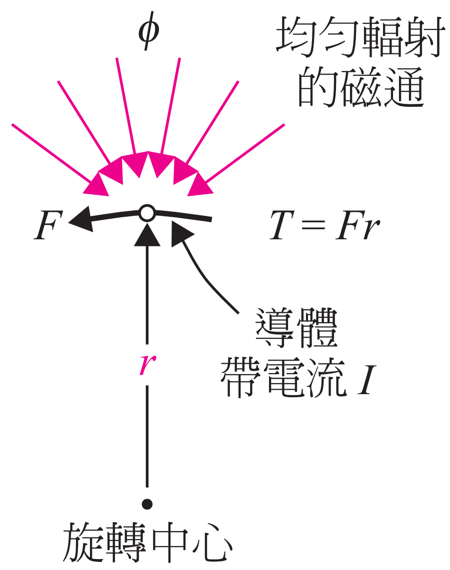


圖 4-64 直流馬達中轉矩的產生

5. 反電動勢和軸速度之間的關係：

當導體在磁場內移動時，在其端點也會產生電壓。此電壓與軸速度成正比，會阻止電流流動，稱為反電動勢 **(back emf)**。

$$e_b = K_m \phi \omega_m \quad \textbf{(4-201)}$$

其中， e_b 代表反電動勢 **(伏特)**，ω_m 為馬達的軸速度 **(rad/sec)**。

※ 永磁式直流馬達的基本分類

1. 直流馬達的磁場可由線圈磁場或永久磁鐵產生。
2. 永磁式直流馬達可根據整流器和電樞的設計加以分類。

   傳統直流馬達有機械電刷和整流器，而有些直流馬達是用電子方式完成整流，此型馬達稱為無刷直流馬達 **(brushless dc motor)**。

   根據電樞結構，永磁式直流馬達可分為三種電樞設計型式：鐵芯 **(iron-core)**、平面繞組 **(surface-wound)** 及動圈式 **(moving-coil)** 馬達。

※ 鐵芯永磁式直流馬達

1. 鐵芯永磁式直流馬達的轉子與定子結構如圖 **4-65** 所示。

2. 具有較高的轉子慣量 (因為旋轉部份包含了電樞線圈)、高電感、低成本及較高的可靠性。

※ 平面繞組直流馬達

1. 圖 **4-66** 所示為平面繞組永磁式直流馬達的轉子結構。

2. 轉子上並無槽隙，所以電樞上無「鑲齒 **(cogging effect)**」效應。

3. 電感量低

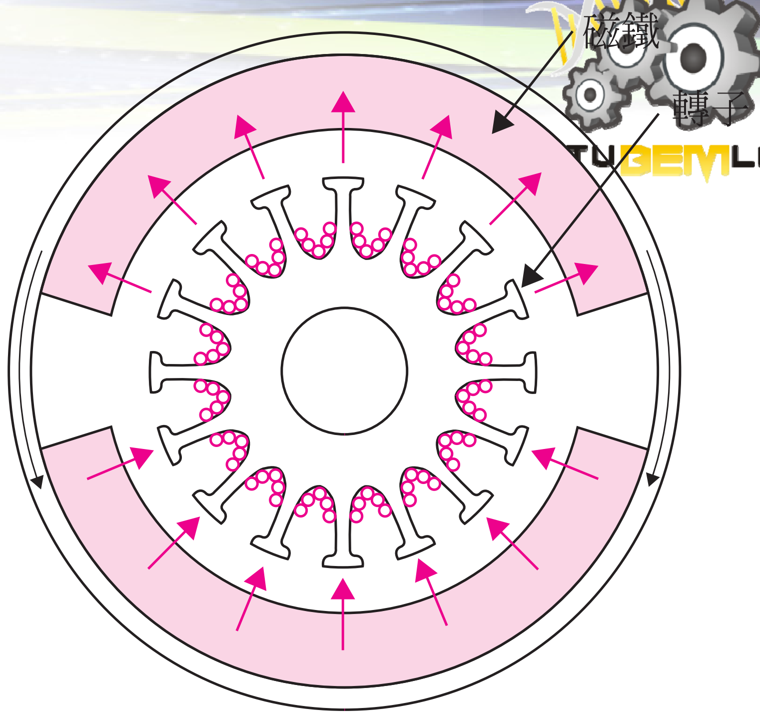


圖 **4-65** 永久磁鐵型鐵芯直流馬達的截面圖

※ 動圈式直流馬達

1. 動圈式馬達設計成使其有極小慣量及極低的電樞電感。其方式是將電樞導體安置於固定磁通路徑和永久磁鐵結構之間的氣隙，如圖 **4-67** 所示。

2. 馬達的橫截面側視圖如圖 **4-68** 所示。

3. 電感很低，通常小於 **100** μ**H**。

4. 所有非必要的元件均由動圈式馬達的電樞中除去，所以慣量很小。

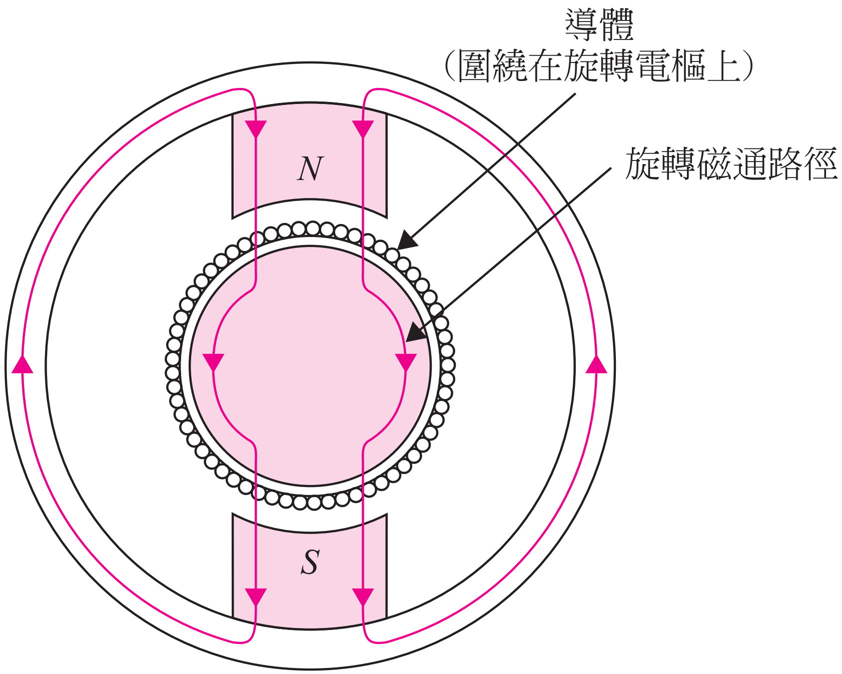


圖 4-66 平面繞組永久磁鐵直流馬達的截面圖

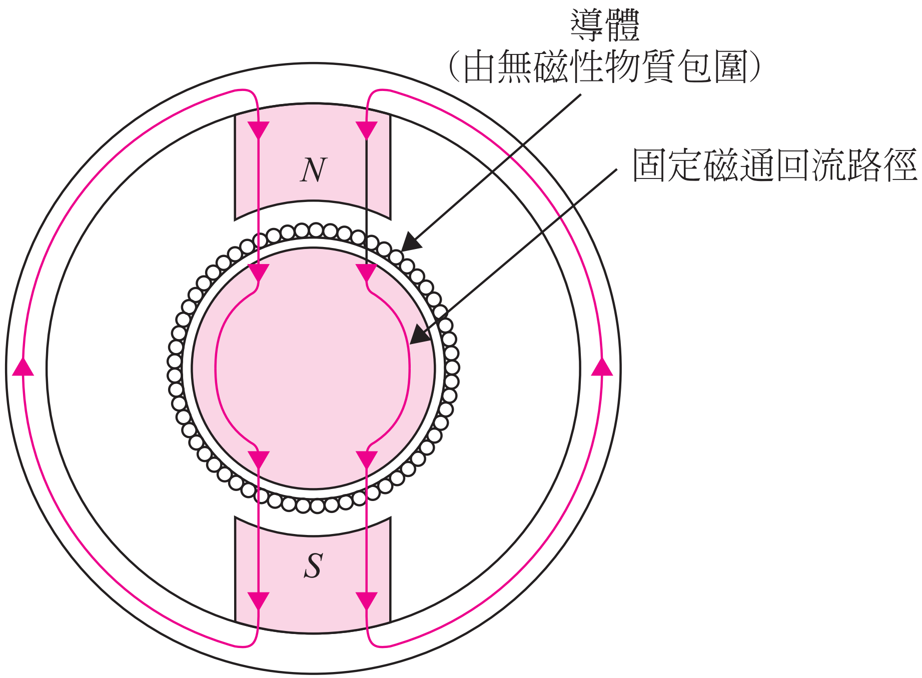


圖 4-67 動圈式永久磁鐵直流馬達的截面圖

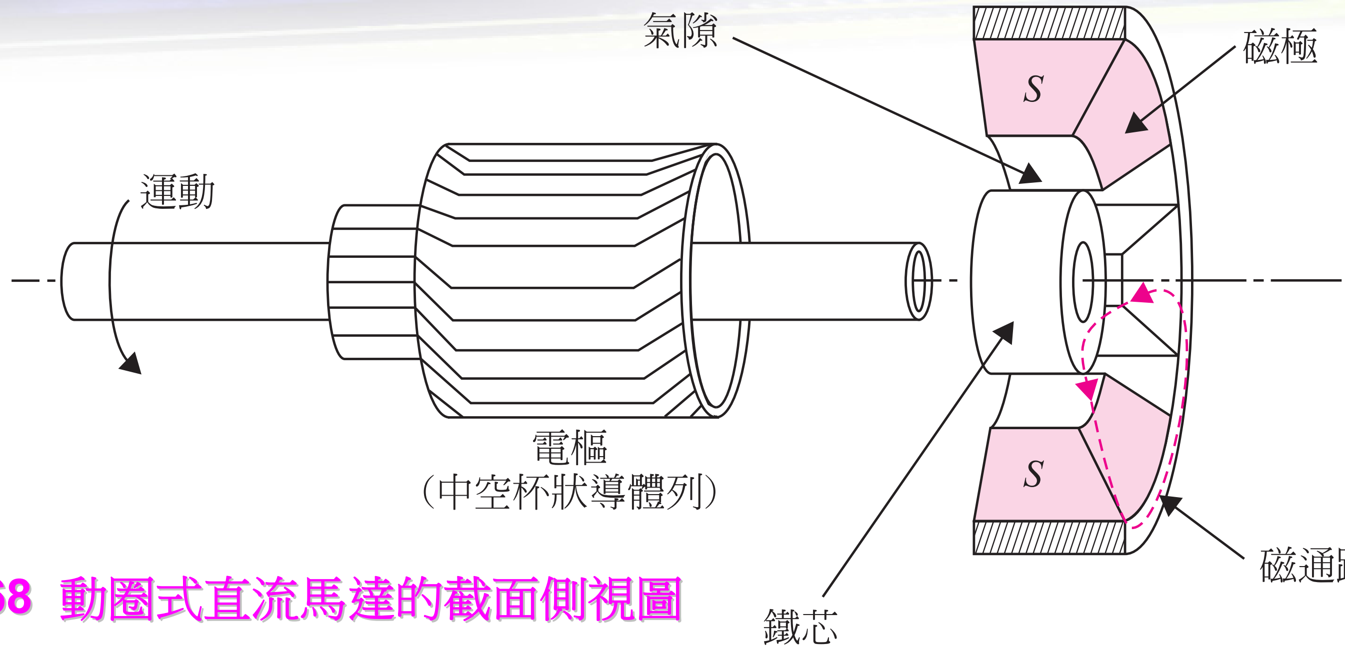


圖 4-68 動圈式直流馬達的截面側視圖

※ 無刷直流馬達

1. 不同於前面所介紹的，無刷直流馬達主要在於電樞電流使用電子的 (而非機械的) 整流器。
2. 最常使用的架構：馬達的轉子由磁鐵組成，並由「背鐵」支撐著；其整流線圈置於旋轉部位的外部，如圖 4-69 所示。
3. 無刷直流馬達可使用於低扭力的慣量應用，如應用於電腦上高性能磁碟機的軸傳動。

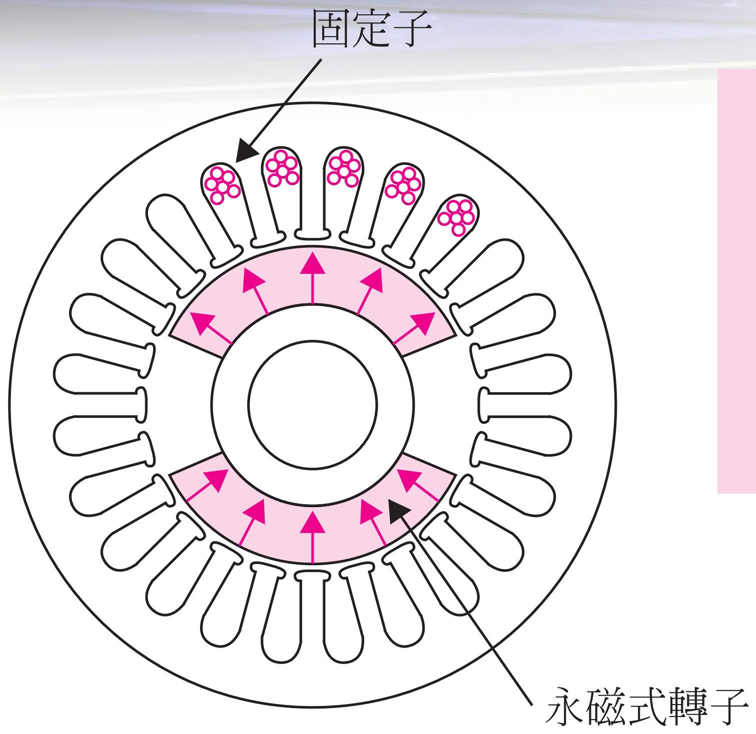


圖 4-69 永磁式無刷直流馬達的截面圖

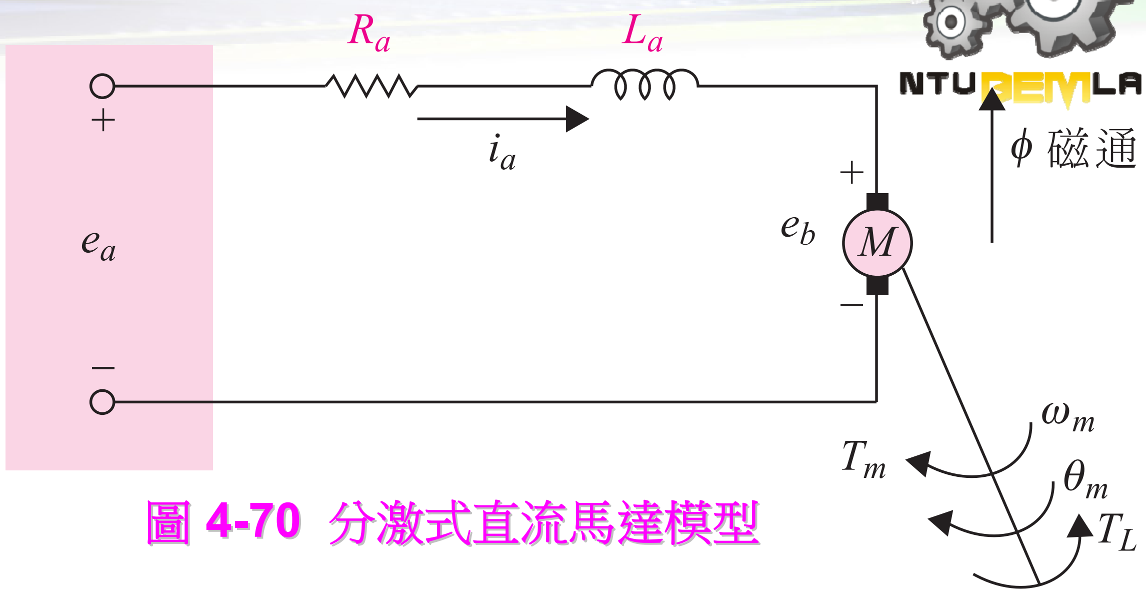


圖 4-70 分激式直流馬達模型

※ 直流馬達的數學模式

1. 永磁式直流馬達的等效電路圖：圖 4-70。
2. 電樞的模式為一電阻 R_a 與一電感 L_a 串聯的電路，電壓源 e_b 代表轉子旋轉時電樞中產生的電壓 (反電動勢)。

3. 馬達的變數和參數：

$i_a(t)$ = 電樞電流	L_a = 電樞電感
R_a = 電樞電阻	$e_a(t)$ = 電樞電壓
$e_b(t)$ = 反電動勢	K_b = 反電動勢常數
$T_L(t)$ = 負載轉矩	ϕ = 氣隙磁通
$T_m(t)$ = 馬達轉矩	$\omega_m(t)$ = 轉子角速度
$\theta_m(t)$ = 轉子角位移	J_m = 馬達轉子慣量
K_i = 轉矩常數	B_m = 黏滯摩擦係數

4. 假設馬達的轉矩與氣隙磁通及電樞電流成正比：

$$T_m(t) = K_m(t)\phi i_a(t) \quad \textbf{(4-202)}$$

$$\Rightarrow T_m(t) = K_i i_a(t) \quad \textbf{(4-203)}$$

因為 ϕ 為常數

K_i 為轉矩常數，單位為 N-m/A，lb-ft/A，或 oz-in./A。

5. 系統因果方程式： $e_a(t)$ 視為眾因之源

$$\frac{di_a(t)}{dt} = \frac{1}{L_a}e_a(t) - \frac{R_a}{L_a}i_a(t) - \frac{1}{L_a}e_b(t) \quad \textbf{(4-204)}$$

$$T_m(t) = K_i i_a(t) \quad \textbf{(4-205)}$$

$$e_b(t) = K_b\frac{d\theta_m(t)}{dt} = K_b\omega_m(t) \quad \textbf{(4-206)}$$

$$\frac{d^2\theta_m(t)}{dt^2} = \frac{1}{J_m}T_m(t) - \frac{1}{J_m}T_L(t) - \frac{B_m}{J_m}\frac{d\theta_m(t)}{dt} \quad \textbf{(4-207)}$$

T

$T_L(t)$ 代表負載摩擦轉矩，如庫倫摩擦

6. 系統的狀態變數可定義為 $i_a(t)$、$\omega_m(t)$ 及 $\theta_m(t)$。

7. 直流馬達系統的狀態方程式：

$$\begin{bmatrix} \frac{di_a(t)}{dt} \\ \frac{d\omega_m(t)}{dt} \\ \frac{d\theta_m(t)}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{R_a}{L_a} & -\frac{K_b}{L_a} & 0 \\ \frac{K_i}{J_m} & -\frac{B_m}{J_m} & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} i_a(t) \\ \omega_m(t) \\ \theta_m(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{L_a} \\ 0 \\ 0 \end{bmatrix} e_a(t) + \begin{bmatrix} 0 \\ -\frac{1}{J_m} \\ 0 \end{bmatrix} T_L(t) \quad \textbf{(4-208)}$$

$T_L(t)$ 可視為狀態方程式的第二個輸入

8.系統的SFG圖(狀態圖)：圖 4-71

9. 馬達位移與輸入電壓間的轉移函數

T_L 已設為零

$$\frac{\Theta_m(s)}{E_a(s)} = \frac{K_i}{L_aJ_ms^3 + (R_aJ_m + B_mL_a)s^2 + (K_bK_i + R_aB_m)s} \quad \textbf{(4-209)}$$

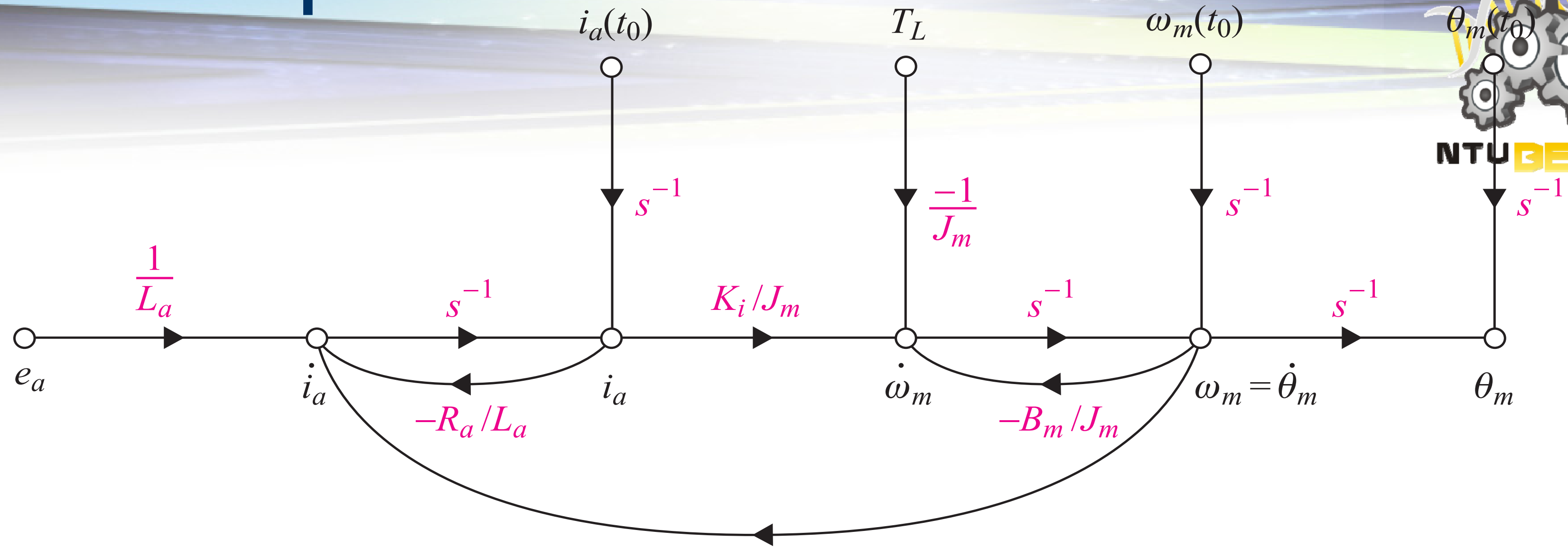


圖 4-71 直流馬達的狀態圖

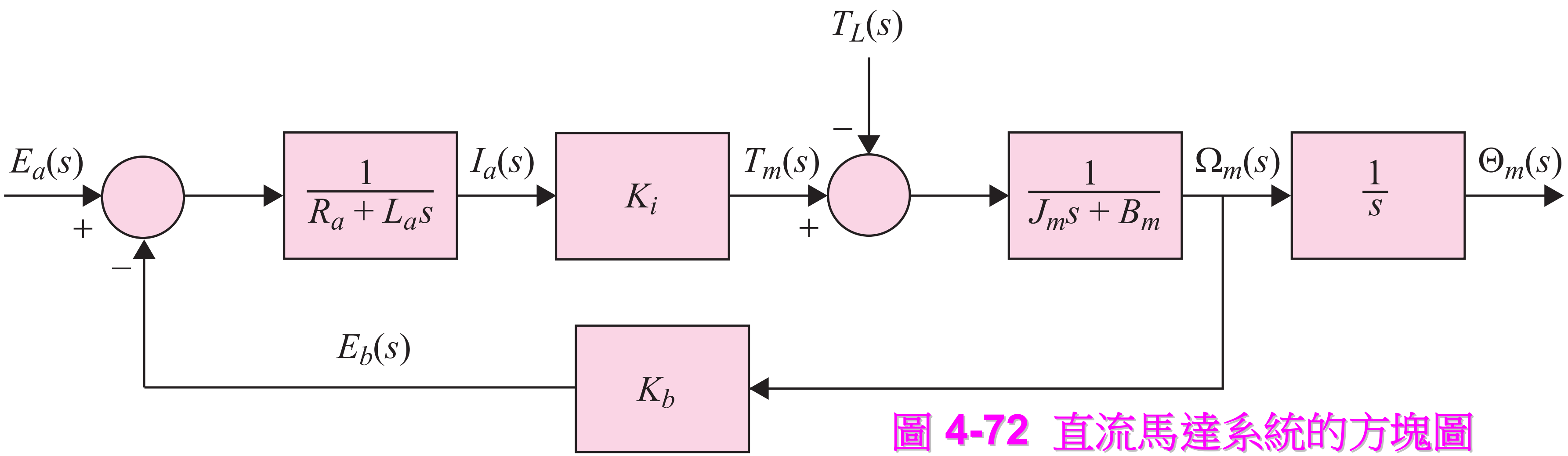


圖 4-72 直流馬達系統的方塊圖

10. 直流馬達系統的方塊圖：圖 **4-72**

- 直流馬達本質上為一積分器

直流馬達本質上為$\Theta_m(s)$與$E_a(s)$兩變數間的積分裝置

因為若 $e_a(t)$ 為固定輸入，則輸出馬達位移的行為如同積分器的輸出；即會隨時間線性增加。

★ 反電動勢

1. 直流馬達本身為一開迴路系統，但圖 **4-71** 的狀態圖和圖 **4-72** 的方塊圖顯示馬達有一由反電動勢產生的「內部」回授迴路。

2. 反電動勢代表回授訊號，其與馬達轉速的負值成正比。

3. 反電動勢效應等於傾向改善馬達穩定性及系統穩定性之「電子摩擦」。

- 反電動勢通常可改善系統的穩定性

※ K_i 與 K_b 間的關係

1. 馬達電樞所發生的機械功率之方程式

$$P = e_b(t)i_a(t) \quad \textbf{(4-210)}$$

機械功率可表示成

$$P = T_m(t)\omega_m(t) \quad \textbf{(4-211)}$$

$T_m(t)$ 之單位在公制系統為 **N-m**，$\omega_m(t)$ 之單位為 **rad/sec**

$$P = T_m(t)\omega_m(t) = K_b\omega_m(t)\frac{T_m(t)}{K_i} \qquad \textbf{(4-212)}$$

$$K_b\,(\text{V/rad/sec}) = K_i\,(\text{N-m/A}) \qquad \textbf{(4-213)}$$

在公制系統中，若 K_b 以 V/rad/sec，K_i 以 N-m/A 表示時，則 K_b 和 K_i 之值相等。

2. (4-210) 式以馬力 **(hp)** 表示

$$P = \frac{e_b(t)i_a(t)}{746}\ \text{hp} \qquad \textbf{(4-214)}$$

以轉矩和角速度來表示，***P*** 可寫成

$$P = \frac{T_m(t)\,\omega_m(t)}{550}\ \text{hp} \qquad \textbf{(4-215)}$$

$T_m(t)$ 之單位為 ft-lb，
$\omega_m(t)$ 之單位為 rad/sec

$$\frac{K_b\omega_m(t)T_m(t)}{746K_i} = \frac{T_m(t)\omega_m(t)}{550} \qquad \textbf{(4-216)}$$

利用 **(4-205)** 式和 **(4-206)** 式，並令上述 **(4-214)** 與 **(4-215)** 兩式相等

$$K_b = \frac{746}{550}K_i = 1.356K_i \qquad \textbf{(4-217)}$$

K_b 之單位為 V/rad/sec，K_i 為 ft-lb/A

※ 具有傳遞延遲的系統

1. 許多系統都會遇到時間延遲問題，尤其是在具有水力、氣動或機械的傳輸系統中。

2. 圖 4-73 的例子，顯然具有傳輸落後或時間延遲。圖 4-74(a) 所示的裝置是將兩種不同的液體以適當的比例混合。

若混合溶液的流動率是每秒 **v** 吋，而 **d** 是混合點和測量點之間的距離，則時間的延遲為

$$T_d = \frac{d}{v} \text{ 秒} \qquad \textbf{(4-218)}$$

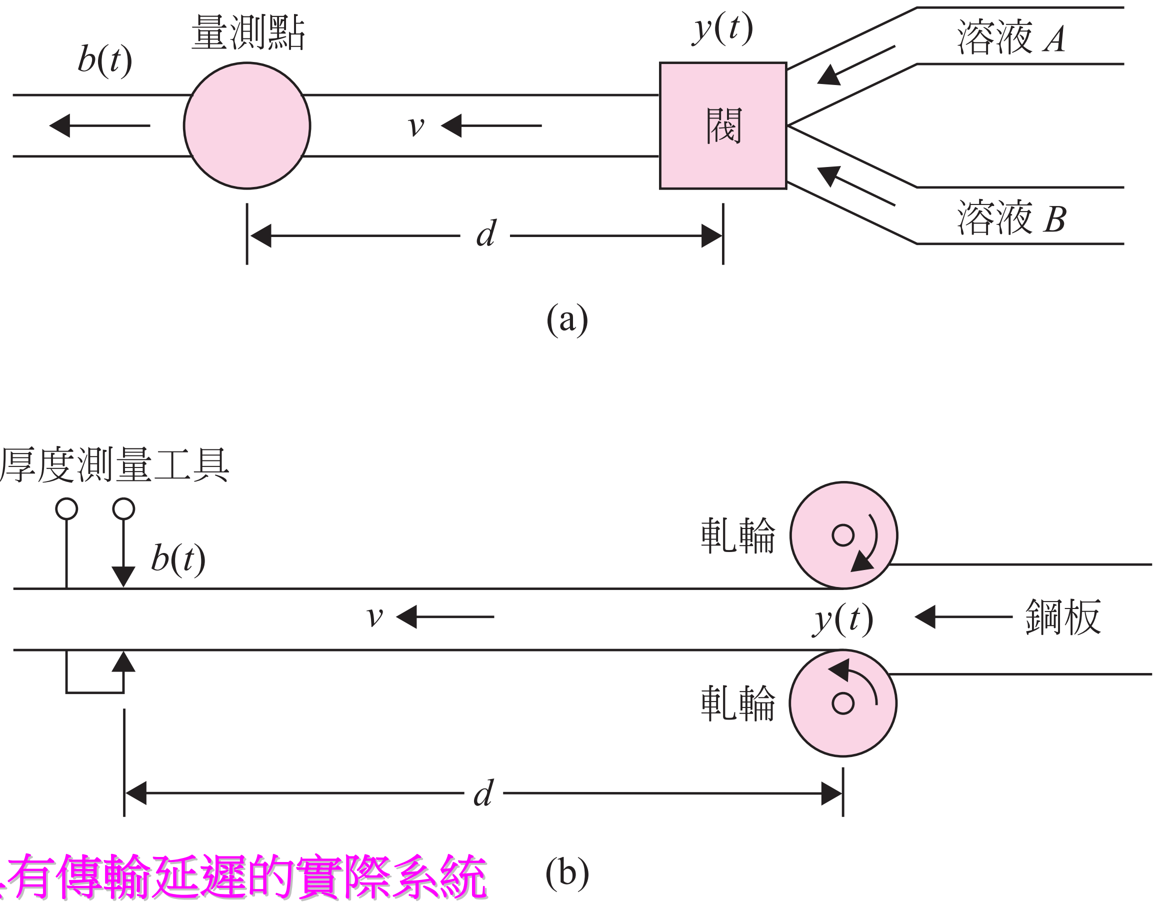


圖 4-73 具有傳輸延遲的實際系統

3. 假設在混合點的濃度是 $y(t)$，則 T_d 秒後才能在監視點量到此值；量測值為

$$b(t) = y(t - T_d) \quad \textbf{(4-219)}$$

$$\Rightarrow \quad B(s) = e^{-T_d s}Y(s) \quad \textbf{(4-220)}$$

4. $b(t)$ 和 $y(t)$ 之間的轉移函數

$$\frac{B(s)}{Y(s)} = e^{-T_d s} \quad \textbf{(4-221)}$$

在圖 4-73(b) 中的裝置是軋鋼板的厚度控制。如同上例一樣，在軋輪和量測點之間厚度的轉移函數可以 **(4-221)** 式來表示。

※ 以有理函數來近似時間延遲函數

超越函數 $e^{-T_d s}$ 近似成一有理函數

1. 馬克勞林級數近似法

$$e^{-T_d s} \cong 1 - T_d s + \frac{T_d^2 s^2}{2} \quad \textbf{(4-222)}$$

或 $$e^{-T_d s} \cong \frac{1}{1 + T_d s + T_d^2 s^2/2} \quad \textbf{(4-223)}$$

2. Pade 近似法 [5, 6]

$$e^{-T_d s} \cong \frac{1 - T_d s/2}{1 + T_d s/2} \quad \textbf{(4-224)}$$

此近似法在轉移函數中引進了一個 s 平面右半平面的零點，因而此近似系統的步階響應在 $t = 0$ 附近會產生一小的負超越量。

※ 非線性系統的線性化

許多實際裝置具有嚴重的非線性特性。對於這些裝置，嚴格來說，其線性化模式只在一很小的操作區間中有效，因此只需求出其操作點之線性化模式。更重要的是，當非線性系統在操作點線性化後，其線性模式可能包含時變元件。

※ 利用Taylor Series的線性化：Classical Representation

- In general, Taylor series may be used to expand a nonlinear function $f(x(t))$ about a reference or operating value $x_0(t)$.
- An operating value could be the equilibrium position in a spring-mass-damper, a fixed voltage in an electrical system, steady state pressure in a fluid system, and so on.
- A function $f(x(t))$ can therefore be represented in a form

$$f(x(t)) = \sum_{i=1}^{n} c_i \left(x(t) - x_0(t)\right)^i \qquad \textbf{(4-225)}$$

where the constant c_i represents the derivatives of $f(x(t))$ with respect to $x(t)$ and evaluated at the operating point $x_0(t)$. That is

$$c_i = \frac{1}{i!}\frac{d^i f(x_0)}{dx^i} \qquad \textbf{(4-226)}$$

Or

$$f(x(t)) = f(x_0(t)) + \frac{df(x_0)}{dx}\left(x(t)-x_0(t)\right) + \frac{1}{2}\frac{d^2 f(x_0)}{dx^2}\left(x(t)-x_0(t)\right)^2 + \frac{1}{6}\frac{d^3 f(x_0)}{dx^3}\left(x(t)-x_0(t)\right)^3 + \cdots + \frac{1}{n!}\frac{d^n f(x_0)}{dx^n}\left(x(t)-x_0(t)\right)^n \qquad \textbf{(4-227)}$$

- If $\Delta(x) = x(t) - x_0(t)$ is small, the series eq. (4-227) converges, and a linearization scheme may be used by replacing $f(x(t))$ with first two terms in Eq. (4-227). That is,

$$\begin{aligned} f(x(t)) &\approx f(x_0(t)) + \frac{df(x_0)}{dx}\left(x(t)-x_0(t)\right) \\ &= c_0 + c_1\Delta x \end{aligned} \qquad \textbf{(4-228)}$$

※ Linearization Using the State Space Approach

1. 代表非線性系統的向量-矩陣狀態方程式：

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{f}[\mathbf{x}(t),\ \mathbf{r}(t)] \qquad \textbf{(4-229)}$$

x(*t*) 代表 ***n*** × **1** 狀態向量，**r(*t*)** 代表 ***p*** × **1** 輸入向量，而 **f[x(*t*)**，**r(*t*)]** 代表 ***n*** × **1** 函數向量。通常，**f** 為狀態向量和輸入向量的函數。

2. 非線性狀態方程式的例子

$$\frac{dx_1(t)}{dt} = x_1(t) + x_2^2(t) \qquad \textbf{(4-230)}$$

$$\frac{dx_2(t)}{dt} = x_1(t) + r(t) \qquad \textbf{(4-231)}$$

3. 線性化的過程是將非線性狀態方程式對正規操作點或軌跡作泰勒級數展開。超過一階以上的泰勒級數項均省略，非線性狀態方程式的線性近似結果發生於正規點。

4. 以 $\mathbf{x}_0(t)$ 代表對應於正規輸入 $\mathbf{r}_0(t)$ 及一些固定的起始狀態的正規操作軌跡。(4-229) 式的非線性狀態方程式在 $\mathbf{x}(t) = \mathbf{x}_0(t)$ 附近展開為泰勒級數，並省略所有高階項，可得

$i = 1, 2, \dots, n$

$$\dot{x}_i(t) = f_i(\mathbf{x}_0,\mathbf{r}_0) + \sum_{j=1}^{n} \frac{\partial f_i(\mathbf{x},\mathbf{r})}{\partial x_j}\bigg|_{\mathbf{x}_0,\mathbf{r}_0}(x_j - x_{0j}) + \sum_{j=1}^{p} \frac{\partial f_i(\mathbf{x},\mathbf{r})}{\partial r_j}\bigg|_{\mathbf{x}_0,\mathbf{r}_0}(r_j - r_{0j}) \qquad (4\text{-}232)$$

令

$$\Delta x_i = x_i - x_{0i} \qquad (4\text{-}233)$$

以及

$$\Delta r_j = r_j - r_{0j} \qquad (4\text{-}234)$$

⇨

$$\Delta \dot{x}_i = \dot{x}_i - \dot{x}_{0i} \qquad (4\text{-}235)$$

因為

$$\dot{x}_{0i} = f_i(\mathbf{x}_0,\mathbf{r}_0) \qquad (4\text{-}236)$$

(4-232) ⇨

$$\Delta \dot{x}_i = \sum_{j=1}^{n} \frac{\partial f_i(\mathbf{x},\mathbf{r})}{\partial x_j}\bigg|_{\mathbf{x}_0,\mathbf{r}_0}\Delta x_j + \sum_{j=1}^{p} \frac{\partial f_i(\mathbf{x},\mathbf{r})}{\partial r_j}\bigg|_{\mathbf{x}_0,\mathbf{r}_0}\Delta r_j \qquad (4\text{-}237)$$

(4-237) 式可寫成向量-矩陣型式：

$$\Delta \dot{\mathbf{x}} = \mathbf{A}^*\Delta \mathbf{x} + \mathbf{B}^*\Delta r \qquad (4\text{-}238)$$

其中

$$\mathbf{A}^* = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \cdot & \cdot & \cdots & \cdot \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \quad \textbf{(4-239)}$$

$$\mathbf{B}^* = \begin{bmatrix} \frac{\partial f_1}{\partial r_1} & \frac{\partial f_1}{\partial r_2} & \cdots & \frac{\partial f_1}{\partial r_p} \\ \frac{\partial f_2}{\partial r_1} & \frac{\partial f_2}{\partial r_2} & \cdots & \frac{\partial f_2}{\partial r_p} \\ \cdot & \cdot & \cdots & \cdot \\ \frac{\partial f_n}{\partial r_1} & \frac{\partial f_n}{\partial r_2} & \cdots & \frac{\partial f_n}{\partial r_p} \end{bmatrix} \quad \textbf{(4-240)}$$

- **補充例題 (第八版)**

  圖 4-S1 為一具有飽和非線性系統的方塊圖。系統的狀態方程式為

$$\dot{x}_1(t) = f_1(t) = x_2(t) \quad \textbf{(4-S1)}$$

$$\dot{x}_2(t) = f_2(t) = u(t) \quad \textbf{(4-S2)}$$

其中飽和非線性的輸入輸出關係可表示為

$$u(t) = (1 - e^{-K|x_1(t)|})\text{SGN } x_1(t) \quad \textbf{(4-S3)}$$

$$\text{SGN } x_1(t) = \begin{cases} +1 & x_1(t) > 0 \\ -1 & x_1(t) < 0 \end{cases} \quad \textbf{(4-S4)}$$

試求線性化的狀態方程式。

<Sol.>

將 **(4-S3)** 式代入 **(4-S2)** 式並利用 **(4-237)** 式，可得線性化的狀態方程式

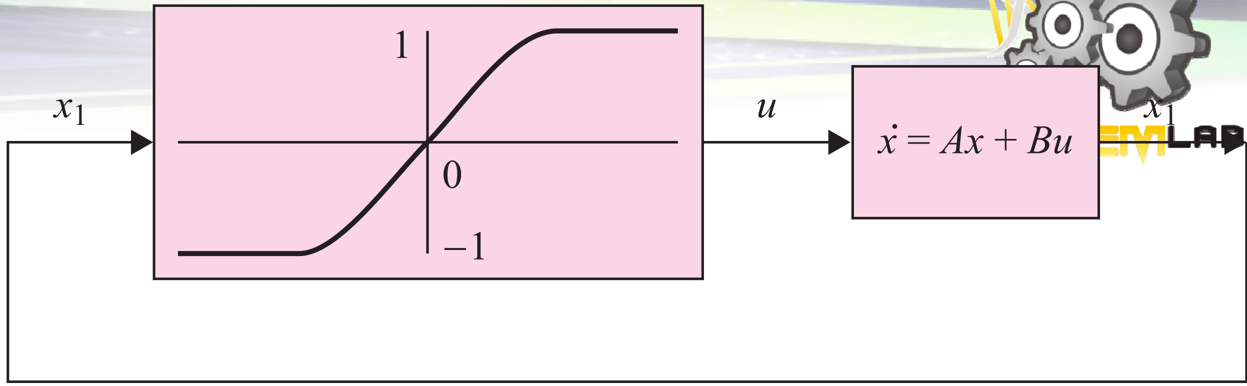


圖 4-S-1 非線性控制系統

x_{01} 代表 $x_1(t)$ 的正規值

$$\Delta\dot{x}_1(t) = \frac{\partial f_1(t)}{\partial x_2}\Delta x_2(t) = \Delta x_2(t) \qquad \textbf{(4-S5)}$$

$$\Delta\dot{x}_2(t) = \frac{\partial f_2(t)}{\partial x_1(t)}\Delta x_1(t) = Ke^{-K|x_{01}|}\Delta x_1(t) \qquad \textbf{(4-S6)}$$

向量矩陣形式

$$\begin{bmatrix} \Delta\dot{x}_1(t) \\ \Delta\dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a & 0 \end{bmatrix} \begin{bmatrix} \Delta x_1(t) \\ \Delta x_2(t) \end{bmatrix} \qquad \textbf{(4-S7)}$$

其中

$$a = Ke^{-K|x_{01}|} = \text{常數} \qquad \textbf{(4-S8)}$$

★ 檢查線性化的正確性

若 x_{01} 是選擇位於非線性特性的原點，$x_{01} = 0$，則 $a = K$；**(4-S6)** 式變成

$$\Delta\dot{x}_2(t) = K\Delta x_1(t) \qquad \textbf{(4-S9)}$$

此線性的模式是等於一具有常數增益 K 的線性放大器

反之，若 x_{01} 是一個大的數字，正規操作點將位於非線性特性的飽和區，而 $a = 0$。這意味著在 $x_1(t)$ [小的 $\Delta x_1(t)$] 中的任何小的變量實際上不會在 $\Delta \dot{x}_2(t)$ 中產生變化。

★ 通常，非線性系統的線性化是指將其變為線性時變系統。

▸ 例題 4-9-1 Find the equation of motion of a pendulum with a mass *m* and a massless rod of a length *l*, as shown in **Fig. 4.74**.

<Sol.>

- Assume the mass is moving in the positive direction as defined by angle θ. Note that θ is measured from the *x* axis in counter-clockwise direction.
- The first step is to draw the free-body diagram of the components of the system, i.e., mass and rod, as shown in **Fig. 4-74(b)**. For the mass m, the equations of motion are

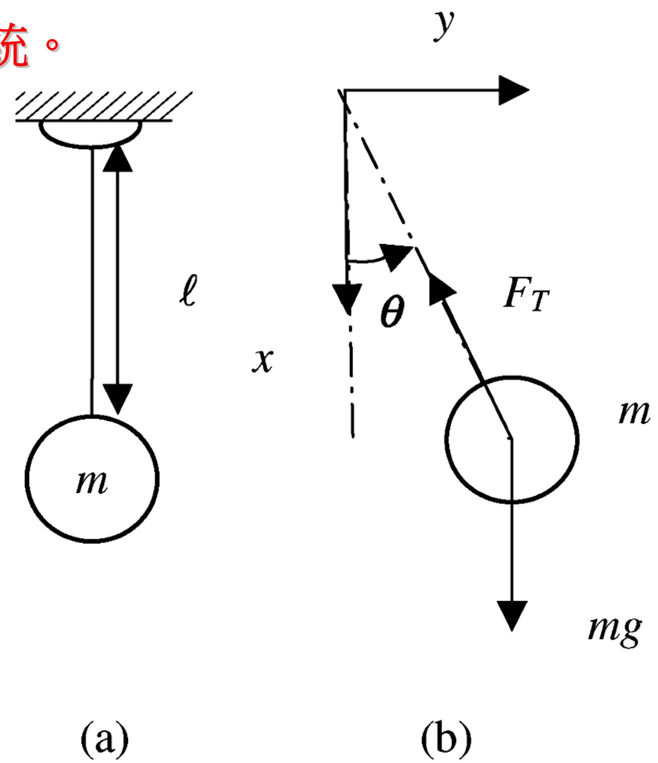


Figure 4-74 (a) A spring-supported pendulum. (b) Free-body diagram of mass *m*.

$$\sum F_x = ma_x \qquad \textbf{(4-241)}$$

$$\sum F_y = ma_y \qquad \textbf{(4-242)}$$

where F_x and F_y are the external forces applied to mass *m*, and a_x and a_y are the components of acceleration of mass *m* in *x* and *y*, respectively.

- Acceleration of mass *m* is a vector with tangential and centripetal components. Using the rectangular coordinate frame (*x*, *y*), acceleration vector is

$$\mathbf{a} = \left(-\ell\ddot{\theta}\sin\theta - \ell\dot{\theta}^2\sin\theta\right)\mathbf{i} + \left(-\ell\ddot{\theta}\cos\theta - \ell\dot{\theta}^2\sin\theta\right)\mathbf{j} \qquad \textbf{(4-243)}$$

where **i** and **j** are unit vectors along *x* and *y* directions, respectively.
As a result,

$$a_x = -\ell\ddot{\theta}\sin\theta - \ell\dot{\theta}^2\sin\theta \qquad \textbf{(4-244)}$$

$$a_y = -\ell\ddot{\theta}\cos\theta - \ell\dot{\theta}^2\sin\theta \qquad \textbf{(4-245)}$$

- Considering the external forces applied to mass, we have

$$\sum F_x = -F_T\cos\theta + mg \qquad \textbf{(4-246)}$$

$$\sum F_y = -F_T\sin\theta \qquad \textbf{(4-247)}$$

- Eqs.(4-241) and (4-242) may therefore be rewritten as

$$-F_T\cos\theta + mg = m\left(-\ell\ddot{\theta}\sin\theta - \ell\dot{\theta}^2\sin\theta\right) \qquad \textbf{(4-248)}$$

$$-F_T\sin\theta = m\left(-\ell\ddot{\theta}\cos\theta - \ell\dot{\theta}^2\sin\theta\right) \qquad \textbf{(4-249)}$$

- Premultiplying Eq. (4-248) by (–sinθ) and Eq. (4-249) by (cosθ) and adding the two, we get

$$-mg\sin\theta = m\ell\ddot{\theta} \qquad \textbf{(4-250)}$$

$$\sin^2\theta + \cos^2\theta = 1$$

After rearranging, Eq. (2-250) is rewritten as

$$m\ell\ddot{\theta} + mg\sin\theta = 0 \qquad \textbf{(4-251)}$$

$$\Rightarrow \quad \ddot{\theta} + \frac{g}{\ell}\sin\theta = 0 \qquad \textbf{(4-252)}$$

- In brief, using static equilibrium position $\theta = 0$ as the operating point, for small motions the linearization of the system implies $\Delta\theta = \theta \approx \sin\theta$. Hence, the linear representation of the system is

$$\ddot{\theta} + \frac{g}{\ell}\theta = 0$$

- Alternatively in the state space form, we define $x_1 = \theta$ and $x_2 = \theta'$ as state variables, and as a result the state space representation of Eq. (4-252) becomes

$$\dot{x}_1 = x_2 \qquad \dot{x}_2 = -\frac{g}{\ell}\sin x_1 \tag{4-253}$$

Substituting Eq. (4-253) into (4-237) with **r**(t) = 0, since there is no input (or external excitations) in this case, we get

$$\Delta\dot{x}_1(t) = \frac{\partial f_1(t)}{\partial x_2}\Delta x_2(t) = \Delta x_2(t) \tag{4-254}$$

$$\Delta\dot{x}_2(t) = \frac{\partial f_2(t)}{\partial x_1(t)}\Delta x_1(t) = -\frac{g}{\ell}\Delta x_1(t) \tag{4-255}$$

where $\Delta x_1(t)$ and $\Delta x_2(t)$ denote nominal values of $x_1(t)$ and $x_2(t)$, respectively.

- Notice that the last two equations are linear and are valid only for small signals. In vector-matrix form, these linearized state equations are written as

$$\begin{bmatrix} \Delta\dot{x}_1(t) \\ \Delta\dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a & 0 \end{bmatrix} \begin{bmatrix} \Delta x_1(t) \\ \Delta x_2(t) \end{bmatrix} \quad \textbf{(4-256)}$$

where

$$a = -\frac{g}{\ell} = \text{constant} \quad \textbf{(4-257)}$$

- It is of interest to check the significance of the linearization. If x_{01} is chosen to be at the origin of the nonlinearity, $x_{01} = 0$, then $a = K$; Eq. (4-255) becomes

$$\Delta\dot{x}_2(t) = K\Delta x_1(t) \quad \textbf{(4-258)}$$

Switching back to classical representation, we get

$$\ddot{\theta} + K\theta = 0 \quad \textbf{(4-259)}$$

▸ **例題 4-9-2** For the pendulum shown in **Fig. 4-74**, re-derive the differential equation using the moment equation.

<Sol.>

- The free-body diagram for the moment equation is shown in **Fig. 4-74**. Applying the moment equation about the fixed point *O*,

$$\sum M_o = m\ell\alpha$$
$$-\ell\sin\theta\cdot mg = m\ell^2\ddot{\theta} \qquad \textbf{(4-260)}$$

This is the same result obtained previously.

- Rearranging the equation in the standard input-output differential equation form,

$$m\ell^2\ddot{\theta} + mg\ell\sin\theta = 0 \qquad \textbf{(4-261)} \quad \Rightarrow \quad \ddot{\theta} + \frac{g}{\ell}\sin\theta = 0 \qquad \textbf{(4-262)}$$

- For small motion, as **Example 4-9-1**,

$$\sin\theta \approx \theta \qquad \textbf{(4-263)}$$

The linearized differential equation is

$$\ddot{\theta} + \omega_n^2\sin\theta = 0 \qquad \textbf{(4-264)}$$

where

$$\omega = \sqrt{\frac{g}{\ell}} \qquad \textbf{(4-265)}$$

▸ 例題 4-9-3 考慮下列的非線性系統：

$$\dot{x}_1(t) = \frac{-1}{x_2^2(t)} \quad \textbf{(4-266)}$$

$$\dot{x}_2(t) = u(t)x_1(t) \quad \textbf{(4-267)}$$

將這些方程式在軌跡 $[x_{01}(t), x_{02}(t)]$ 附近線性化，而此值就是這些方程式在起始條件 $x_1(0) = x_2(0) = 1$ 和輸入 $u(t) = 0$ 時的解答。

<Sol.>

1. 將 (4-267) 式兩邊對 *t* 作積分，針對以上的條件可以得到

$$x_2(t) = x_2(0) = 1 \quad \textbf{(4-268)}$$

(4-266) 式 ⇨ $x_1(t) = -t + 1$ **(4-269)**

(4-266) 式及 (4-267) 式線性化所要的軌跡可用下列式子表示：

$$x_{01}(t) = -t + 1 \quad \textbf{(4-270)}$$

$$x_{02}(t) = 1 \quad \textbf{(4-271)}$$

2. 計算 (4-237) 式的係數：

$$\frac{\partial f_1(t)}{\partial x_1(t)} = 0 \quad \frac{\partial f_1(t)}{\partial x_2(t)} = \frac{2}{x_2^3(t)} \quad \frac{\partial f_2(t)}{\partial x_1(t)} = u(t) \quad \frac{\partial f_2(t)}{\partial u(t)} = x_1(t) \qquad \textbf{(4-272)}$$

(4-237) 式 ⇨

$$\Delta \dot{x}_1(t) = \frac{2}{x_{02}^3(t)}\Delta x_2(t) \qquad \textbf{(4-273)}$$

$$\Delta \dot{x}_2(t) = u_0(t)\Delta x_1(t) + x_{01}(t)\Delta u(t) \qquad \textbf{(4-274)}$$

3. 線性化後的方程式可寫成

$$\begin{bmatrix}\Delta \dot{x}_1(t)\\ \Delta \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}0 & 2\\ 0 & 0\end{bmatrix}\begin{bmatrix}\Delta x_1(t)\\ \Delta x_2(t)\end{bmatrix} + \begin{bmatrix}0\\ 1-t\end{bmatrix}\Delta u(t) \qquad \textbf{(4-275)}$$

(4-270) 及 (4-271) 式代入 (4-273) 及 (4-274) 式

此為一組具有時變係數的線性狀態方程式。

▸ 例題4-9-4

圖 4-75 為一磁浮球系統。此系統的目標在於利用輸入電壓 ***e*(*t*)** 來調整電磁鐵中的電流以控制鋼球的位置。系統的微分方程式為：

$$M\frac{d^2y(t)}{dt^2} = Mg - \frac{i^2(t)}{y(t)} \qquad \textbf{(4-276)}$$

$$e(t) = Ri(t) + L\frac{di(t)}{dt} \qquad \textbf{(4-277)}$$

其中
$e(t)$ = 輸入電壓
$i(t)$ = 線圈電流
L = 線圈電感
g = 動加速度
$y(t)$ = 球之位置
R = 線圈電阻
M = 球之質量

針對平衡點 $y_0(t) = x_{01}$ = 常數，試將系統線性化。

<Sol.>

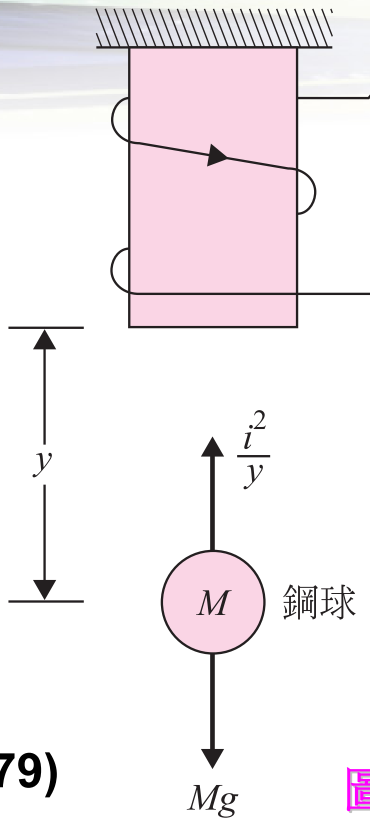
1. 定義狀態變數為 $x_1(t) = y(t)$、$x_2(t) = dy(t)/dt$，及 $x_3(t) = i(t)$

2. 系統的狀態方程式

$$\frac{dx_1(t)}{dt} = x_2(t) \qquad \textbf{(4-278)}$$

$$\frac{dx_2(t)}{dt} = g - \frac{1}{M}\frac{x_3^2(t)}{x_1(t)} \qquad \textbf{(4-279)}$$

$$\frac{dx_3(t)}{dt} = -\frac{R}{L}x_3(t) + \frac{1}{L}e(t) \qquad \textbf{(4-280)}$$



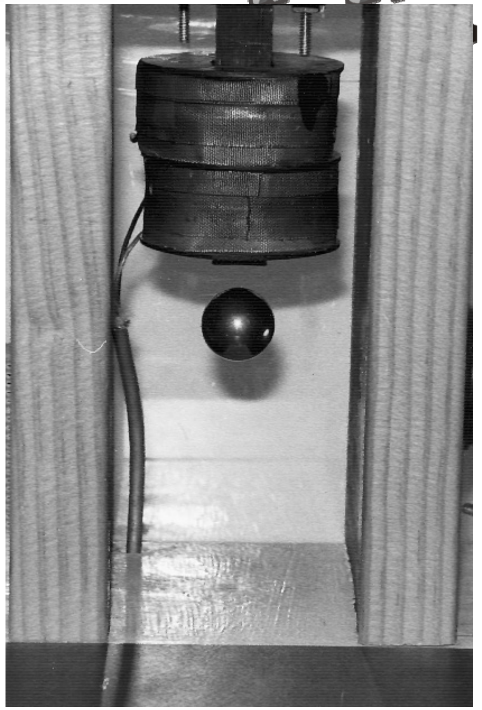


圖 4-75 磁浮球系統

3. 針對平衡點 $y_0(t) = x_{01}$ = 常數，將系統線性化

$$x_{02}(t) = \frac{dx_{01}(t)}{dt} = 0 \qquad \textbf{(4-281)} \qquad\qquad \frac{d^2y_0(t)}{dt^2} = 0 \qquad \textbf{(4-282)}$$

4. *i*(*t*) 的額定值

將 (4-282) 式代入 (4-276) 式而得到

$$i_0(t) = x_{03}(t) = \sqrt{Mgx_{01}} \quad \textbf{(4-284)}$$

5. 線性化後的狀態方程式可表程向量-矩陣型式： $\Delta\dot{\mathbf{x}} = \mathbf{A}^*\Delta\mathbf{x} + \mathbf{B}^*\Delta r$

係數矩陣 **A***及 **B*** 為

$$\mathbf{A}^* = \begin{bmatrix} 0 & 1 & 0 \\ \dfrac{x_{03}^2}{Mx_{01}^2} & 0 & \dfrac{-2x_{03}}{Mx_{01}} \\ 0 & 0 & -\dfrac{R}{L} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \dfrac{g}{x_{01}} & 0 & -2\left(\dfrac{g}{Mx_{01}}\right)^{1/2} \\ 0 & 0 & -\dfrac{R}{L} \end{bmatrix} \quad \textbf{(4-285)}$$

$$\mathbf{B}^* = \begin{bmatrix} 0 \\ 0 \\ \dfrac{1}{L} \end{bmatrix} \quad \textbf{(4-286)}$$

※ ANALOGIES

- With the analogy, mass *M* and inertia *J* are analogous to inductance *L*, the spring constant *K* is analogous to the inverse of capacitance 1/*C*, and the viscous-friction coefficient *B* is analogous to resistance *R*.

▸ **例題4-10-1**

It is a logical, in **Example 4-1-1**, to assign *v*(*t*), the velocity, and *f*(*t*), the force acting on the spring, as state variables, since the former is analogous to current in *L* and the latter is analogous to the voltage across *C*. Writing the force on *M* and the velocity of the spring as functions of the state variables and the input force *f*(*t*), we have

- **Force on mass:**

$$M\frac{dv(t)}{dt} = f_k(t) - Bv(t) + f(t) \qquad \textbf{(4-287)}$$

- **Velocity of spring:**

$$\frac{1}{k}\frac{df_k(t)}{dt} = v(t) \qquad \textbf{(4-288)}$$

以位移***y*(*t*)**表示，整理後可得

$$\frac{d^2y}{dt^2}+\frac{B}{M}\frac{dy(t)}{dt}+\frac{K}{M}y(t)=\frac{f(t)}{M} \qquad \textbf{(4-289)}$$

- 將例題**4-2-1**的**(4-67)**式整理後，可得

$$L\frac{d_i(t)}{dt}=-e_c(t)-Ri(t)+e(t) \qquad \textbf{(4-290)}$$

利用**(4-66)**式的電流關係式，可得

$$C\frac{de_c(t)}{dt}=i(t) \qquad \textbf{(4-291)}$$

- The comparison of Eq. (4-287) with Eq. (4-290) and Eq. (4-288) with Eq. (4-291) clearly shows the analogies among the mechanical and electrical components.

▸ 例題4-10-2

As another example of writing the dynamic equations of a mechanical system with translational motion, consider the system shown in Fig.4-9 (a). Because the spring is deformed when it is subject to a force *f*(*t*), two displacements, y_1 and y_2, must be assigned to the end points of the spring. The free body diagram of the system is shown in Fig 4-9(b).

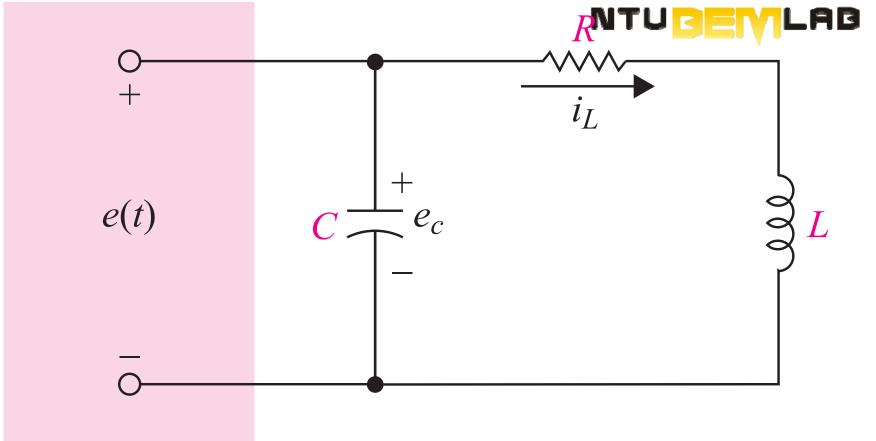


圖 4-76 類比於圖 4-10 機械系統的電網路

- 力方程式

$$f(t) = K[y_1(t) - y_2(t)] \qquad \textbf{(4-292)}$$

$$K[y_1(t) - y_2(t)] = M\frac{d^2 y_2(t)}{dt^2} + B\frac{dy_2(t)}{dt} \qquad \textbf{(4-293)}$$

重新整理可得

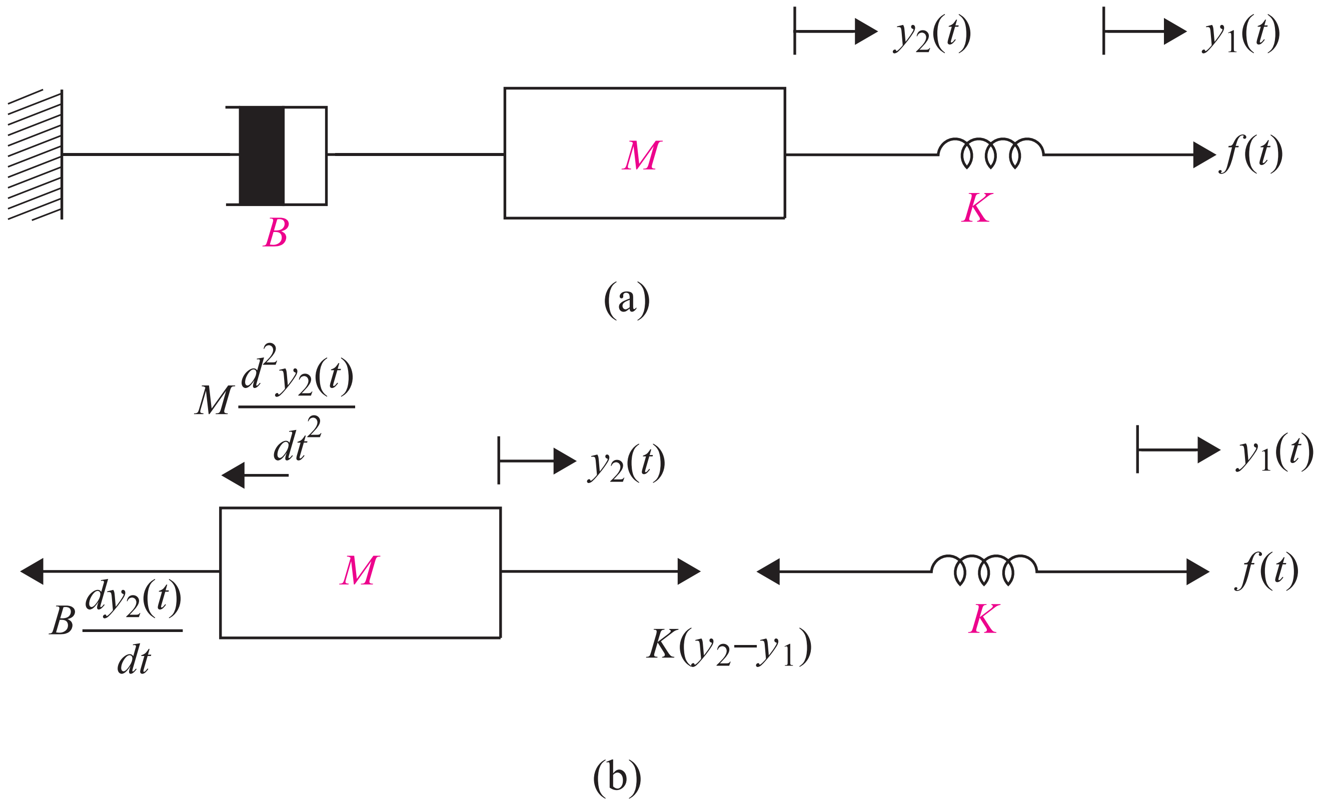


圖 4-9 例題 4-1-2 中的機械系統。(a) 質量-彈簧-摩擦系統；(b) 自由物體圖

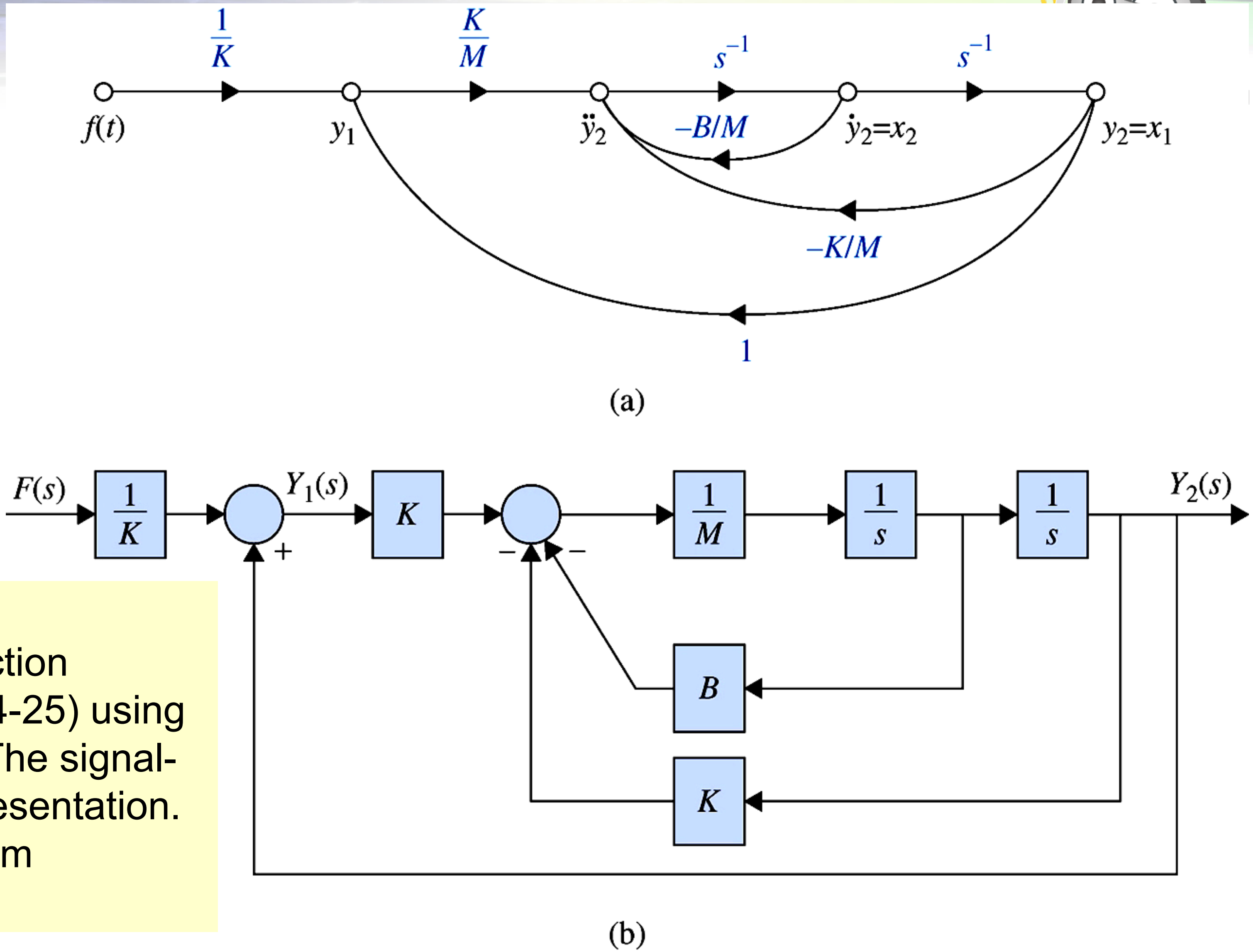


Figure 4-10
Mass-spring-friction system of Eq. (4-25) using Eq. (4-22). (a) The signal-flow graph representation. (b) Block diagram representation.

$$y_1(t) = y_2(t) + \frac{1}{K} f(t)$$ **(4-294)**

$$\frac{d^2 y_2(t)}{dt^2} = -\frac{B}{M}\frac{dy_2(t)}{dt} + \frac{K}{M}[y_1(t) - y_2(t)]$$ **(4-295)**

- 使用**SFG**圖**(Refer to Fig. 4-10(a))**且定義狀態變數

$$x_1(t) = y_2(t)$$

$$x_2(t) = \frac{dy_2(t)}{dt}$$

The state equations are written directly from the state diagram:

$$\frac{dx_1(t)}{dt} = x_2(t)$$ **(4-296)**

$$\frac{dx_2(t)}{dt} = -\frac{B}{M} x_2(t) + \frac{1}{M} f(t)$$ **(4-297)**

★ 另一種狀態變數設定：

將物體 ***M*** 的速度 ***v*(*t*)** 設計為一個狀態變數，而加於彈簧的力 ***f***$f_k(t)$ 當作另一個狀態變數。

$$\frac{dv(t)}{dt} = -\frac{B}{M}v(t) + \frac{1}{M}f_k(t) \qquad \textbf{(4-298)}$$

$$f_k(t) = f(t) \qquad \textbf{(4-299)}$$

為何有兩個狀態變數 $v(t)$ 和 $f_k(t)$，但卻只有一個狀態方程式 **(4-298)?**

說明

- One may wonder why there is only one state equation in Eq. (4-287), where there are two state variables in *v*(*t*) and *f*(*t*).
- The two state equations of Eqs. (4-296) and (4-297) clearly show that the system is of the second order.
- The situation is better explained by referring to the analogous electric network of the system shown in Fig 4-76. Although the network has two energy-storage elements in *L* and *C*, and thus there should be two state variables, the voltage across the capacitance in this case is redundant, since it is equal to the applied voltage *e*(*t*).
- Eqs. (4-298) and (4-297) can provide only the solutions to the velocity of *M*, *v*(*t*), which is the same as $dy_2(t)/dt$, once *f*(*t*) is specified.

- Then $y_2(t)$ is determined by integrating $v(t)$ with respect to t.
- The displacement $y1(t)$ is then found using Eq. (4-292). On the other hand, Eqs. (4-296) and (4-297) give the solutions to $y_2(t)$ and $dy_2(t)/dt$ directly, and $y_1(t)$ is obtained from Eq. (4-292).

利用此系統的類比電網路(圖 4-76)

電容器是「多餘」的元件，因為 $e_c(t)$ 等於外加電壓 $e(t)$。

當 $f(t)$ 為已知時

解出 → $v(t)$[亦即 $dy_2(t)/dt$]

解出 → $y_2(t) = \int v(t)dt$

解出 → $y_1(t) = y_2(t) + f(t)/K$

★ 系統的轉移函數

直接對圖 4-10(a) 的狀態圖使用增益公式求得

$$\frac{Y_2(s)}{F(s)} = \frac{1}{s(Ms + B)} \quad (4\text{-}300)$$

$$\frac{Y_1(s)}{F(s)} = \frac{Ms^2 + Bs + K}{Ks(Ms + B)} \quad (4\text{-}301)$$

▸ EXAMPLE 4-10-3 A Pneumatic System

Dry air passes through a valve into a rigid 1 m^3 container, as shown in Fig. 4-77, at a constant temperature T = 25 °C (= 298 K). The pressure at the left-hand side of the valve is p_i, which is higher than the pressure in the tank p. Assuming a laminar flow, the valve resistance becomes linear, R = 200 sec/m^2. Find the time constant of the system.

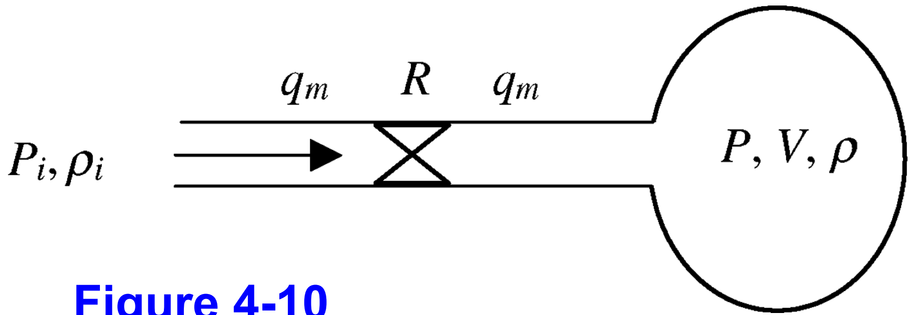


Figure 4-10
A pneumatic system with a valve and a spherical rigid tank.

<Sol.>

- Assuming air as an ideal gas, isothermal process, and low pressures, from Example 4-5-5, the equation of the system is

$$\frac{RV}{R_{air}T}\dot{p} + p = p_i \qquad \textbf{(4-302)}$$

where air at standard pressure and temperature is represented as an ideal gas,

$$pv = \frac{p}{\rho} = R_{air}T \qquad \rho = \frac{1}{R_{air}T}p \qquad \textbf{(4-303)}$$

- The time constant is

$$\tau = \frac{RV}{R_{air}T} = \frac{(200)(1)}{88.63(298)} = 7.5(10^{-3})\,\text{sec} \qquad \textbf{(4-304)}$$

where, from reference [1] at the end of this chapter,

$$R_{air} = 53.35\frac{\text{ftlb}_f}{\text{lb}_m\,°\text{R}}\frac{0.3048\text{m}}{\text{ft}}\frac{4.45\text{N}}{\text{lb}_f}\frac{\text{kgm/sec}^2}{\text{N}}\frac{\text{lb}_m}{0.4536\text{kg}}\frac{°\text{R}}{°\text{K}(9/5)} = 88.63\frac{\text{m}^2}{\text{sec}^2\,°\text{K}}$$

▸ **EXAMPLE 4-10-4**

A One-Tank Liquid-Level System

For the liquid-level system shown in **Fig. 4-45**, $C = A/g$ is the capacitance and $\rho = R$ is the resistance. As a result, system time constant is $\tau = RC$. Comparing the thermal, fluid, and electrical system, similar analogies may be obtained, as shown in **Table 4-8**.

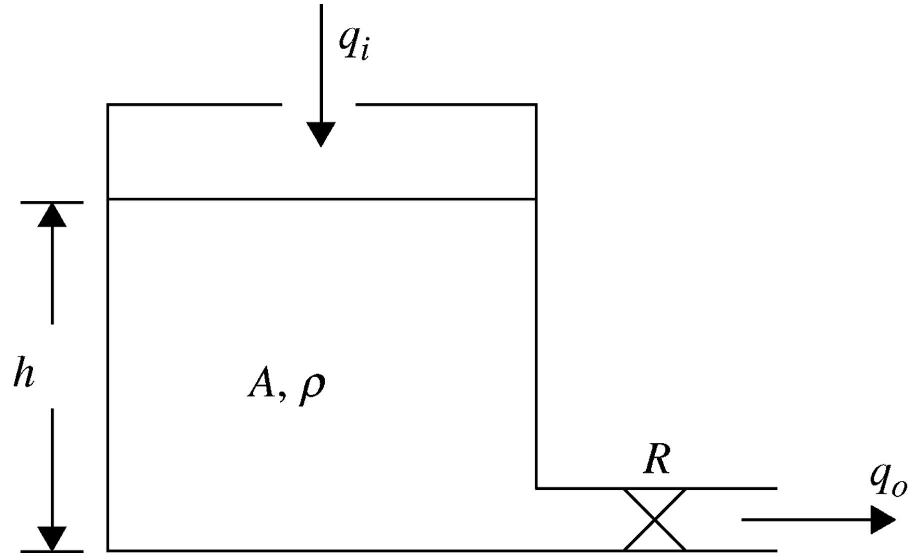


Figure 4-45 A single-tank liquid-level system.

TABLE 4-8 Mechanical, Thermal, and Fluid Systems and Their Electrical Equivalents

System	R, C, L	Analogy
Mechanical (translation)	$F = Bv(t)$ $R = B$ $F = K\int v(t)dt$ $C = \frac{1}{K}$ $v(t) = \frac{1}{M}\int F\,dt$ $L = M$	$e => F$ $i(t) => v(t)$ where $e =$ voltage $i(t) =$ current $F =$ force $v(t) =$ linear velocity
Mechanical (rotation)	$T = B\omega(t)$ $R = B$ $T = K\int \omega(t)dt$ $C = \frac{1}{K}$ $\omega = \frac{1}{J}T\,dt$ $L = J$	$e => T$ $i(t) => \omega(t)$ where $e =$ voltage $i(t) =$ current $T =$ torque $\omega(t) =$ angular velocity
Fluid (incompressible)	$\Delta P = Rq(t)$ (laminar flow) R depends on flow regime $q(t) = C\dot{P}$ C depends on flow regime $L = \frac{\rho\ell}{A}$ (flow in a pipe)	$e => \Delta P$ $i(t) => q(t)$ where $e =$ voltage $i(t) =$ current

(Continued)

TABLE 4-8 (Continued)

System	R, C, L	Analogy
	where $A =$ area of cross section $l =$ length $\rho =$ fluid density	$P =$ pressure $q(t) =$ volume flow rate
Thermal	$R = \dfrac{\Delta T}{q}$ $T = \dfrac{1}{C}\int q\,dt$	$e => T$ $i(t) => q(t)$ where $e =$ voltage $i(t) =$ current $T =$ temperature $q(t) =$ heat flow

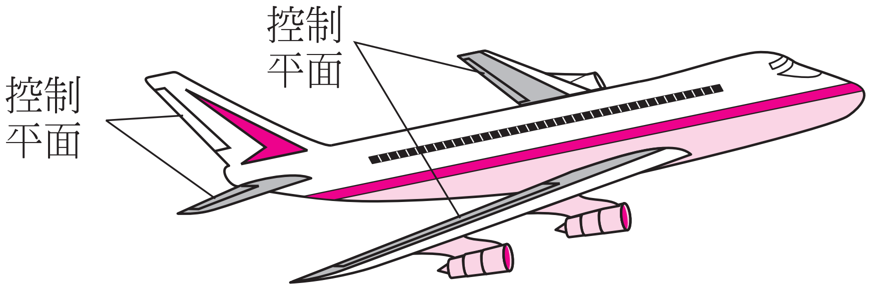
※ CASE STUDIES

▸ EXAMPLE 4-11-1

Consider the system in **Fig. 4-78**. The purpose of the system considered here is control the positions of the fins of a modern airship. Due to the requirements of improved response and reliability, the surfaces of modern aircraft are controlled by electric actuators with electronic controls. Gone are the days when the ailerons, rudder, and elevators of the aircraft were all linked to the cockpit through mechanical linkages. The so-called fly-by-wire control system used in modern aircraft implies that the attitude of aircraft is no longer controlled entirely be mechanical linkages. **Fig. 4-78** illustrates the controlled surfaces and the block diagram of one axis of such a position-control system. **Fig. 4-79** shows the analytical block diagram of the system using the dc-motor model given in **Fig 4-72**. The system is simplified to the extent that saturation of the amplifier gain and motor torque, gear backlash, and shaft compliances have all been neglected. (When you get into the real world, some of these nonlinear effects should be incorporated into the mathematical model to come up with a better controller design that works in reality. The reader should refer to Chapter 6, where these topics are discussed in more detail.)

1. 位置控制系統的控制平面與方塊圖：圖 4-78 。圖 4-79 則為利用放大器/直流馬達(如 圖 4-72 所示)的分析方塊圖。

此簡化系統將放大器增益和馬達轉矩的飽和，齒輪背隙，及軸的撓性忽略而得到。



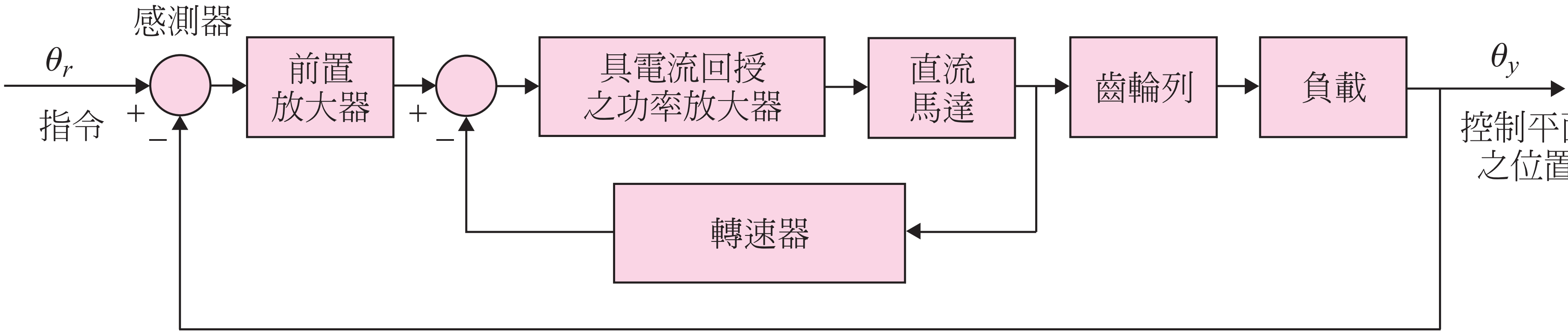


圖 4-78 飛機傾斜度控制系統方塊圖

此系統的目的為系統輸出 $\theta_y(t)$ 跟隨輸入 $\theta_r(t)$ 變化。

各系統參數的值如下：

編碼器增益	$K_s = 1$ V/rad
前置放大器增益	$K =$ 可調整的
功率放大器增益	$K_1 = 10$ V/V
電流回授增益	$K_2 = 0.5$ V/A
轉速計回授增益	$K_t = 0$ V/rad/sec
馬達電樞阻抗	$R_a = 5.0\ \Omega$
馬達電樞電感	$L_a = 0.003$ H
馬達轉矩常數	$K_i = 9.0$ oz-in./A
馬達反電動勢常數	$K_b = 0.0636$ V/rad/sec
馬達轉子慣量	$J_m = 0.0001$ oz-in.-sec^2
負載慣量	$J_L = 0.01$ oz-in.-sec^2
馬達黏滯摩擦係數	$B_m = 0.005$ oz-in.-sec
負載黏滯摩擦係數	$B_L = 1.0$ oz-in.-sec
馬達和負載之間的齒輪列比值	$N = \theta_y / \theta_m = 1/10$

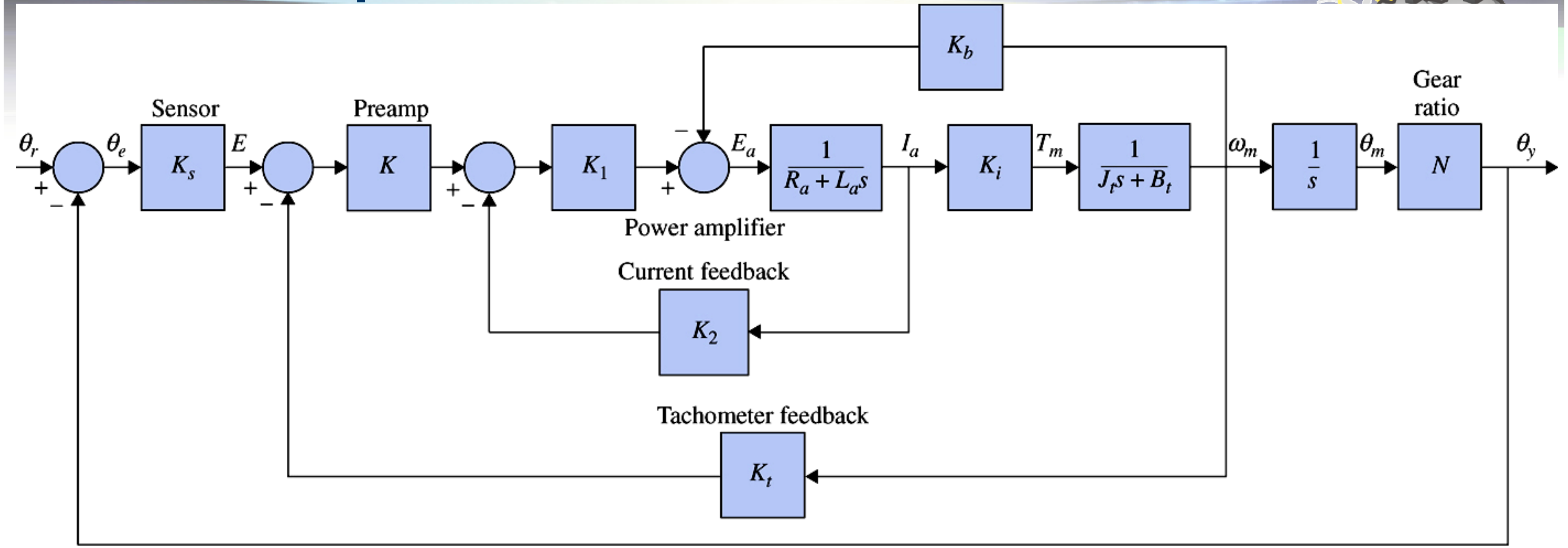


圖 4-79 圖 4-78 之系統轉移函數方塊圖

2. 馬達是經由一齒輪比為 ***N*** 的齒輪列接到負載，$\theta_y = N\theta_m$ 。

3. 從馬達側看到的等效總慣量及黏滯摩擦係數分別為

$$J_1 = J_m + N^2 J_L = 0.0001 + 0.001/100 = 0.0002 \text{ oz-in.-sec}^2$$
$$B_t = B_m + N^2 B_L = 0.005 + 1/100 = 0.015 \text{ oz-in.-sec} \qquad \textbf{(4-305)}$$

4. 單位回授系統的順向路徑轉移函數可利用 **SFG** 增益公式求得：

$$G(s)=\frac{\Theta_y(s)}{\Theta_e(s)}$$

因為 ***G*(*s*)** 的最高次項為 s^3，故此系統為三階系統。

$$=\frac{K_sK_1K_iKN}{s[L_aJ_ts^2+(R_aJ_t+L_aB_t+K_1K_2J_t)s+R_aB_t+K_1K_2B_t+K_iK_b+KK_1K_tK_i]} \quad \textbf{(4-306)}$$

5. 放大器-馬達系統的電氣時間常數為

$$\tau_a=\frac{L_a}{R_a+K_1K_2}=\frac{0.003}{5+5}=0.0003 \quad \textbf{(4-307)}$$

馬達-負載的機械時間常數為

$$\tau_t=\frac{J_t}{B_t}=\frac{0.0002}{0.015}=0.01333 \quad \textbf{(4-308)}$$

由於馬達的低電感，所以電氣時間常數遠小於機械時間常數。因此可以忽略電樞電感 L_a，以便加以近似。

三階系統便近似為二階

6. 順向路徑轉移函數為

$$G(s)=\frac{K_sK_1K_iKN}{s[(R_aJ_t+K_1K_2J_t)s+R_aB_t+K_1K_2B_t+K_iK_b+KK_1K_iK_t]}$$

$$=\frac{\dfrac{K_sK_1K_iKN}{R_aJ_t+K_1K_2J_t}}{s\left(s+\dfrac{R_aB_t+K_1K_2B_t+K_iK_b+KK_1K_iK_t}{R_aJ_t+K_1K_2J_t}\right)} \quad \textbf{(4-S-1)}$$

$$G(s) = \frac{4500K}{s(s + 361.2)} \quad \textbf{(4-S-2)}$$

標準二階系統轉移函數

7. 1) 自然無阻尼頻率：

$$\omega_n = \pm\sqrt{\frac{K_s K_1 K_i K N}{R_a J_t + K_1 K_2 J_t}} = \pm\sqrt{4500K} \quad \text{rad/sec} \quad \textbf{(4-S-3)}$$

2) 阻尼比：

$$\zeta = \frac{R_a B_t + K_1 K_2 B_t + K_i K_b + K K_1 K_i K_t}{2\sqrt{K_s K_1 K_i K N (R_a J_t + K_1 K_2 J_t)}} = \frac{2.692}{\sqrt{K}} \quad \textbf{(4-S-4)}$$

自然無阻尼頻率 ω_n 與放大器增益的平方根 ***K*** 成正比，而阻尼比 ζ 與 ***K*** 成反比。

8. 單位回授控制系統之特性方程式：

$$s^2 + 361.2s + 4500K = 0 \quad \textbf{(4-S-5)}$$

※ 太陽追蹤器系統

1. 模擬太陽追蹤系統，主要目的為控制太空船的傾斜度，以便準確地追蹤太陽的位置。
2. 系統的示意圖：圖 4-80。

   誤差鑑別器的主要元件為裝在一長方型細縫後的兩片矽質光伏電池。

   兩個電池要安裝成感測器指向太陽時，經由細縫的光柱會同時照在兩片電池上。

★ 座標系統 ⇨ 圖 4-81

1. 系統的座標系統中心位於輸出齒輪
2. 參考軸為直流馬達的固定外框，且所有的旋轉量測均根據此軸。太陽軸 (或從輸出齒輪到太陽的直線)與參考軸成一角度 $\theta_r(t)$，且 $\theta_o(t)$ 表示艙體軸與參考軸的夾角。
3. 控制系統的目標在使 $\theta_r(t)$ 和 $\theta_o(t)$ 之間的誤差 $\alpha(t)$ 趨近於零

$$\alpha(t) = \theta_r(t) - \theta_o(t) \quad \textbf{(4-309)}$$

★ 誤差鑑別器

1. 當艙體完全對準太陽時，$\alpha(t) = 0$，且 $i_a(t) = i_b(t) = I$，或 $i_a(t) = i_b(t) = 0$ 。
2. 太陽光與光伏電池的幾何關係

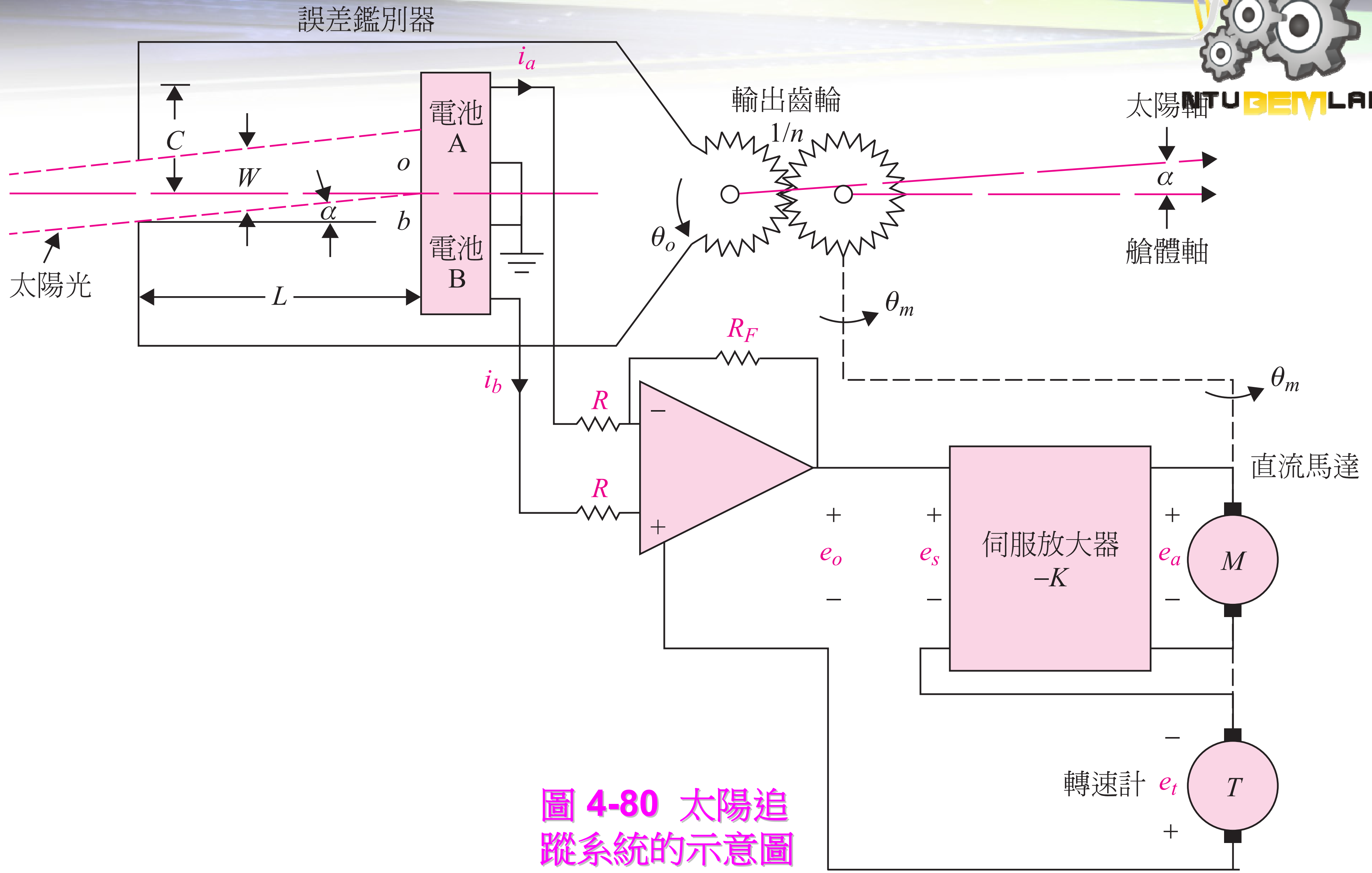


圖 4-80 太陽追蹤系統的示意圖

$$oa = \frac{W}{2} + L\tan\alpha(t) \qquad \textbf{(4-310)}$$

$$ob = \frac{W}{2} - L\tan\alpha(t) \qquad \textbf{(4-311)}$$

參考
圖 4-81

3. 針對一已知的 $\alpha(t)$，oa 代表太陽光照射在電池 A 的寬度，而 ob 則是電池 B 上的寬度。由於電流 $i_a(t)$ 與 oa 成正比，且 $i_b(t)$ 與 ob 成正比，我們可以得到

$$i_a(t) = I + \frac{2LI}{W}\tan\alpha(t) \qquad \textbf{(4-312)}$$

$$i_b(t) = I - \frac{2LI}{W}\tan\alpha(t) \qquad \textbf{(4-313)}$$

$0 \le \tan\alpha(t) \le W/2L$

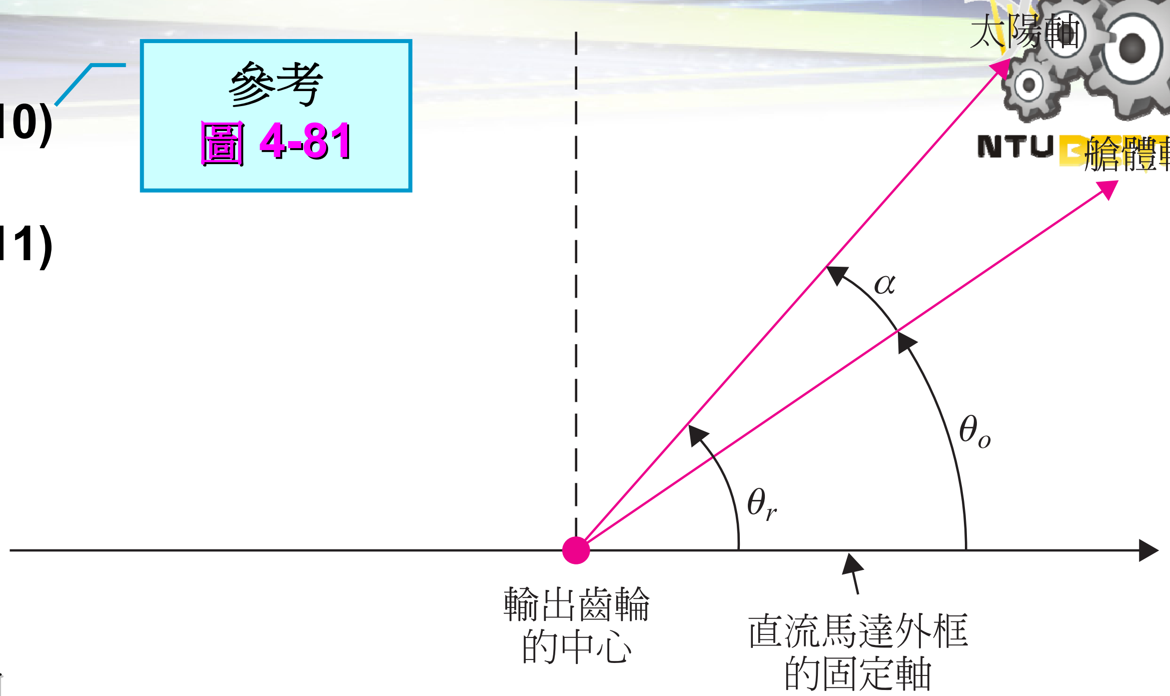


圖 4-81 太陽追蹤系統的座標系統

4. 當 $W/2L \le \tan\alpha(t) \le (C-W/2)/L$ 時，太陽光完全照在電池 A 上，且 $i_a(t) = 2I$，$i_b(t) = 0$。當 $(C-W/2)/L \le \tan\alpha(t) \le (C+W/2)/L$ 時，$i_a(t)$ 則由 $2I$ 到零線性地減少。當 $\tan\alpha(t) \ge (C+W/2)/L$ 時，$i_a(t) = i_b(t) = 0$。

5. 誤差鑑別器可由圖 **4-82** 中的非線性特性表示，其中對於小角度 $\alpha(t)$，$\tan\alpha(t)$ 以 $\alpha(t)$ 近似之。

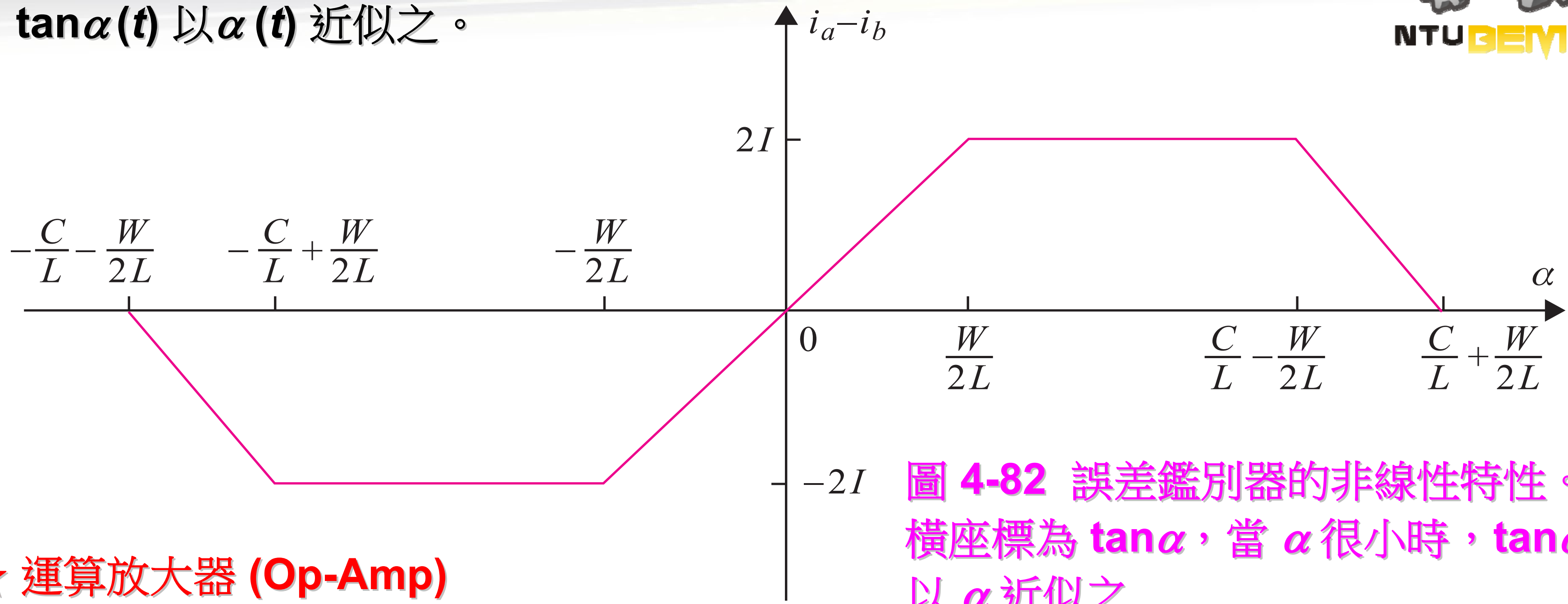


圖 **4-82** 誤差鑑別器的非線性特性。橫座標為 $\tan\alpha$，當 α 很小時，$\tan\alpha$ 以 α 近似之

★ 運算放大器 (Op-Amp)

1. op-amp 輸出與電流 $i_a(t)$ 及 $i_b(t)$ 之間的關係

$$e_o(t) = -R_F[i_a(t) - i_b(t)] \qquad \textbf{(4-314)}$$

★ 伺服放大器

1. 伺服放大器的增益為 $-K$。參考圖 **4-83**，伺服放大器的輸出可表示為

$$e_a(t) = -K[e_o(t) + e_t(t)] = -Ke_s(t) \qquad \textbf{(4-315)}$$

★ 轉速計

1. 轉速計的輸出電壓，e_t，與轉速計常數 K_t 和馬達的角速度相關：

$$e_t(t) = K_t\omega_m(t) \quad \textbf{(4-316)}$$

2. 輸出齒輪的角位移，經由齒輪比，1/*n*，與馬達位置相關。因此

$$\theta_o = \frac{1}{n}\theta_m \quad \textbf{(4-317)}$$

★ 直流馬達

參考4-6 節

1. 直流馬達模型，其方程式為

$$e_a(t) = R_a i_a(t) + e_b(t) \quad \textbf{(4-318)}$$

$$e_b(t) = K_b\omega_m(t) \quad \textbf{(4-319)}$$

$$T_m(t) = K_i i_a(t) \quad \textbf{(4-320)}$$

$$T_m(t) = J\frac{d\omega_m(t)}{dt} + B\omega_m(t) \quad \textbf{(4-321)}$$

J 及 ***B*** 為馬達軸的慣量與黏滯摩擦係數，在 **(4-318)** 式中忽略馬達的電感。

※ 系統方塊圖：圖 **4-83**

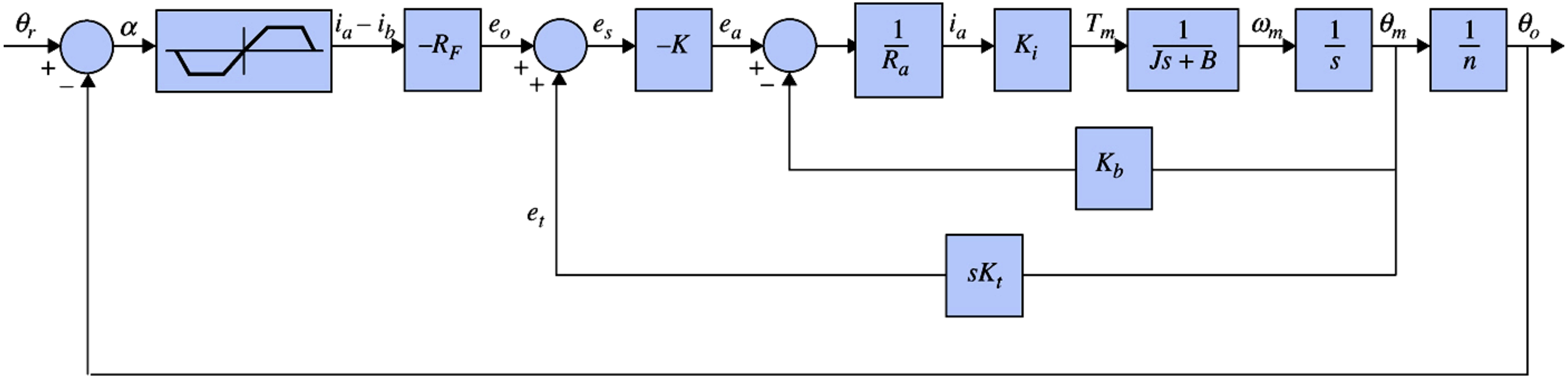


圖 4-83 太陽追蹤系統的方塊圖

▸ **EXAMPLE 4-11-3**

Classically, the quarter-car model is used in the study of vehicle suspension systems and the resulting dynamic response due to various road inputs. Typically, the inertia, stiffness, and damping characteristics of the system as illustrated in **Fig. 4-84(a)** are modeled in a two degree of freedom (2-DOF) system, as shown in **(b)**. Although a 2-DOF system is a more accurate model, it is sufficient for the following analysis to assume a 1-DOF model, as shown in **(c)**.

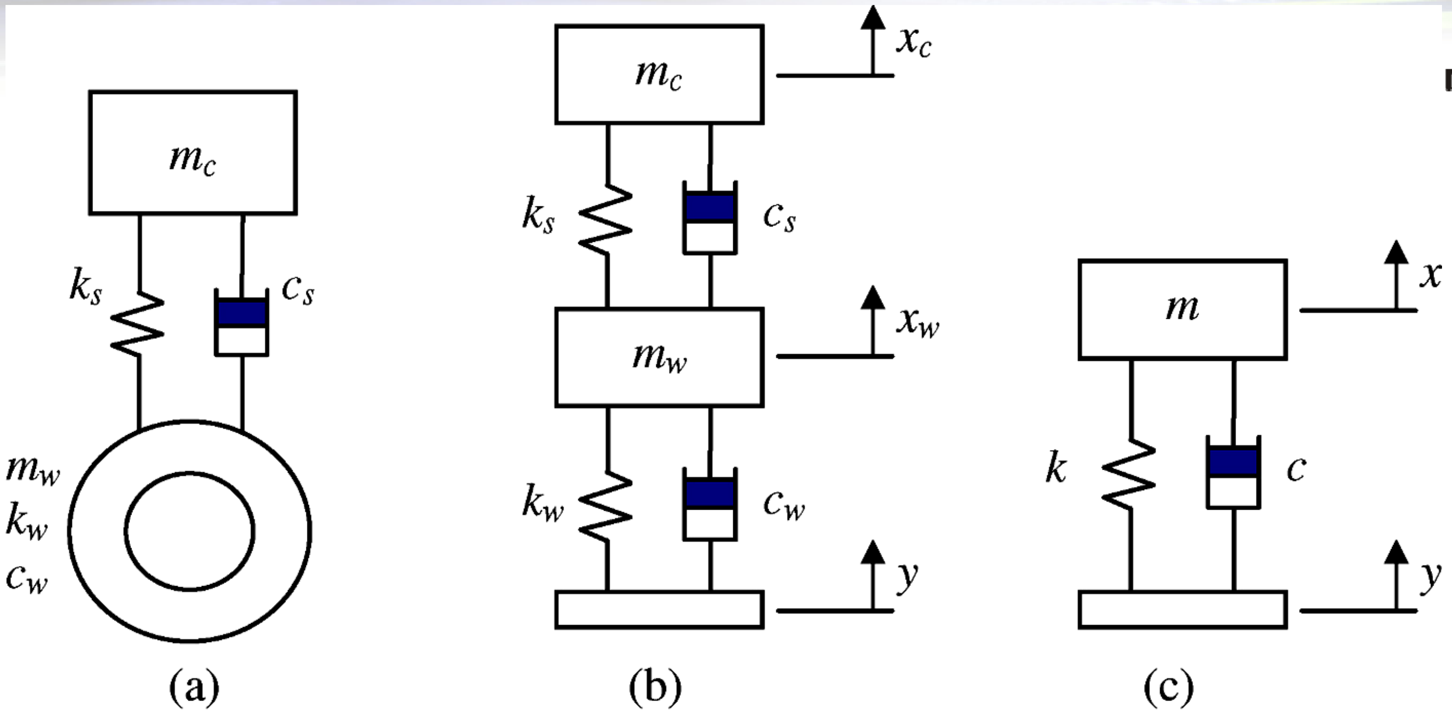


Figure 4-84 Quarter-car model realization. (a) Quarter car. (b) Two degrees of freedom. (c) One degree of freedom.

◆ **Open-Loop Base Excitation**

- Given the system illustrated in **Fig. 4-84(c)**, where

m	Effective ¼ car mass	10 kg
K	Effective stiffness	2.7135 N/m
C	Effective damping	0.9135 N-m/s^{-1}
$x(t)$	Absolute displacement of the mass m	m
$y(t)$	Absolute displacement of the base	m
$z(t)$	Relative displacement $(x(t)-y(t))$	m

- The equation of motion of the system is defined as follows:

$$m\ddot{x}(t)+c\dot{x}(t)+kx(t)=c\dot{y}(t)+ky(t) \qquad \textbf{(4-322)}$$

which can be simplified by substituting the relation $z(t) = x(t) - y(t)$ and non-dimensionalizing the coefficients to the form

$$\ddot{z}(t)+2\zeta\omega\dot{z}(t)+\omega_n^2 z(t)=-\ddot{y}(t)=-a(t) \qquad \textbf{(4-323)}$$

- The Laplace transform of Eq. (4-323) yields the input-output relationship

$$\frac{Z(s)}{A(s)}=\frac{-1}{s^2+2\xi\omega_n s+\omega_n^2} \qquad \textbf{(4-324)}$$

where the base acceleration $A(s)$ is the Laplace transform of $a(t)$ and is the input, and relative displacement $Z(s)$ is the output.

◆ Closed-Loop Position Control

- Active control of the suspension system is to be achieved using the same dc motor described in Section 4-7 used in conjunction with a rack as shown in **Fig. 4-85**.
- In **Fig.4-85**, $T(t)$ is the torque produced by the motor with shaft rotation θ, and r is the radius of the motor drive gear. Thus, Eq. (4-322) is rewritten to include the active component, $f(t)$,

$$m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky + f(t) \qquad \textbf{(4-325)}$$

where

$$m\ddot{z} + c\dot{z} + kz = f(t) - m\ddot{y} = f(t) - ma(t) \qquad \textbf{(4-326)}$$

$$f(t) = \frac{T(t) - (J_m\theta + B_m\theta)}{r} \qquad \textbf{(4-327)}$$

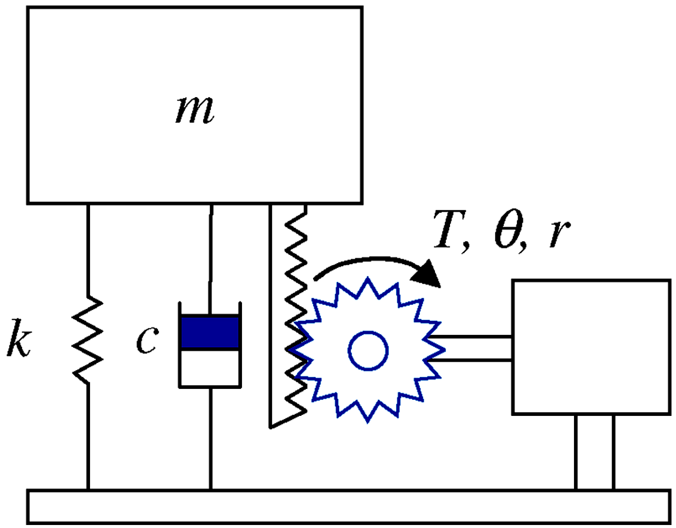


Figure 4-85 Active control of the 1-DOF model via a dc motor and rack.

- Because $z = \theta r$, we can substitute Eq. (4-327) into Eq.(4-326), rearrange, and take the *Laplace transform* to get

$$Z(s) = \frac{r}{(mr^2 + J_m)s^2 + (cr^2 + B_m)s + kr^2}[T(s) - mrA(s)] \qquad \textbf{(4-328)}$$

Noting that

$$Z(s)/r = \Theta(s)$$

this is analogous to previous input-output relationships where

$$\Theta(s) = G_{eq}(T(s) - T_d(s))$$

hence, the term $mrA(s)$ is interpreted as a disturbance torque.

- The block diagram in **Fig. 4-86** can thus be compared to **Fig. 4-85**, where

$J = mr^2 + J_m, B = cr^2 + B_m$ and $k = kr^2$

- Using the principle of superposition, this system is rearranged to the following form:

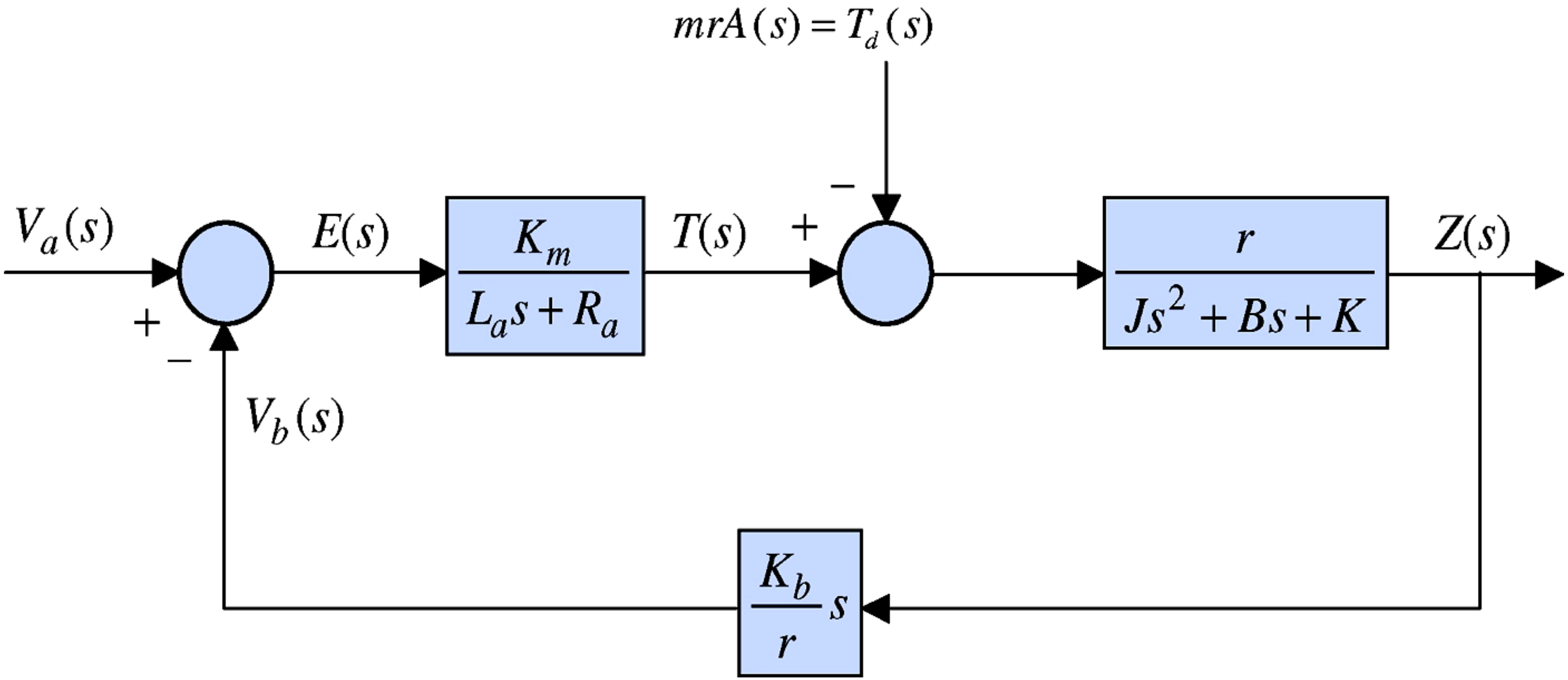


Figure 4-86 Block diagram of an armature-controlled dc motor.

$$Z(s)=\frac{\dfrac{K_m r}{R_a}}{\left(\dfrac{L_a}{R_a}s+1\right)(Js^2+Bs+K)+\dfrac{K_m K_b}{R_a}s}V_a(s)-\frac{\left(\dfrac{L_a}{R_a}s+1\right)r}{\left(\dfrac{L_a}{R_a}s+1\right)(Js^2+Bs+K)+\dfrac{K_m K_b}{R_a}s}mrA(s) \qquad \text{(4-329)}$$

※ MATLAB TOOLS

- Apart form the MATLAB toolboxes appearing with the chapter, this chapter does not contain any software because of its focus on theoretical development.
- In Chapters 6 and 9, where we address more complex control-system modeling and analysis, we will introduce the Automatic Control Systems MATLAB and SIMULINK tools.

Chapter 4 動態系統的模型化

- The Automatic Control Systems software (ACSYS) consists of a number of m-files and GUIs (graphical user interface) for the analysis of simple control engineering transfer functions.
- It can be invoked from the MATLAB command line by simply typing ***Acsys*** and then by clicking on the appropriate pushbutton. A specific MATLAB tool has been developed for most chapters of this textbook. Throughout this chapter, we have identified subjects that may be solved using ACSYS, with a box in the left margin of the text titled "MATLAB TOOL."
- The most relevant components of ACSYS to the problems in this chapter are Virtual Lab and SIMLab, which are discussed in detail in Chapter 6.
- These simulation tools provide the user with virtual experiments and design projects using systems involving dc motors sensors, electronic components, and mechanical components.

Homework:

4-2
4-3
4-4: (c)
4-5: (a)
4-8
4-10
4-13: (d) and (e)
4-14
4-18
4-19
4-21
4-24
4-26
4-27
4-30
4-31
4-38
4-41
4-42
4-44
4-46
4-48
4-49
4-51
4-52
4-54
4-57
4-58
4-59
4-61

請注意：此三題為參考習題，請同學們試試看，不用繳交。

Thank You !